Theory and Practice of Free-Electron Lasers

Particle Accelerator School
Day 1

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Course Content

Course Schedule

	Monday	Tuesday	Wednesday	Thursday	Friday
9:00–10:30	Intro. to FEL Relativistic Dynamics	Optical Architectures	Wigglers	RF Linac	Final Exam
10:45–12:15	1-D FEL Theory	Optical Architectures	Wigglers	RF Linac	Final Exam Lab Report Due
1:15–3:15	Simulation Lab	Simulation Lab	Simulation Lab	Simulation Lab	
3:30–5:30	1-D FEL Theory	Optical Architectures	RF Linac	Electron Injectors	

Chapter 1
Introduction to Free-Electron Lasers

Introduction to Free-Electron Lasers

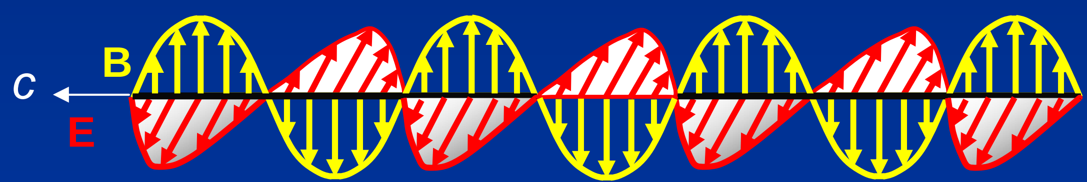
- The nature of light
- Gaussian beam
- Laser beam emittance
- Longitudinal coherence
- How a quantum laser works
- How an FEL works
- Basic features of FEL
- RF-linac FEL
- Fourth-generation Light Sources
- Applications of FEL

Light can be described as both particles (photons) and waves

- Light consists of photons each having energy $\mathcal{E} = h\nu$ where h = Planck's constant (h = 6.626 x 10^{-34} J-s) and ν = frequency of the light; $\nu\lambda = c$
  Photon energy can be calculated from wavelength as follows

$$\mathcal{E} = \frac{1.24eV}{\lambda(\mu)} \qquad \mathcal{E} = \frac{12.4keV}{\lambda(\text{Å})}$$

- Light can also be described as a travelling electromagnetic (EM) wave.



$$c = 2.9979 \times 10^{8}\,\frac{m}{s}$$

We can treat the EM wave as a sinusoidal plane wave. In our convention, the electric field is in the x direction and magnetic field in the y direction. For a wave travelling in the positive z direction, the fields are given below

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}E_0 \cos(kz - \omega t + \varphi)$$
$$\mathbf{B}(z,t) = \hat{\mathbf{y}}B_0 \cos(kz - \omega t + \varphi)$$

where k = wavenumber in m^{-1}
ω= angular frequency in s^{-1}
φ = phase in radians

Gaussian Laser Beam

rms radius in x

$$\sigma_x^2 = \frac{\int_{-\infty}^{\infty} I(x,y)x^2 dxdy}{\int_{-\infty}^{\infty} I(x,y)dxdy}$$

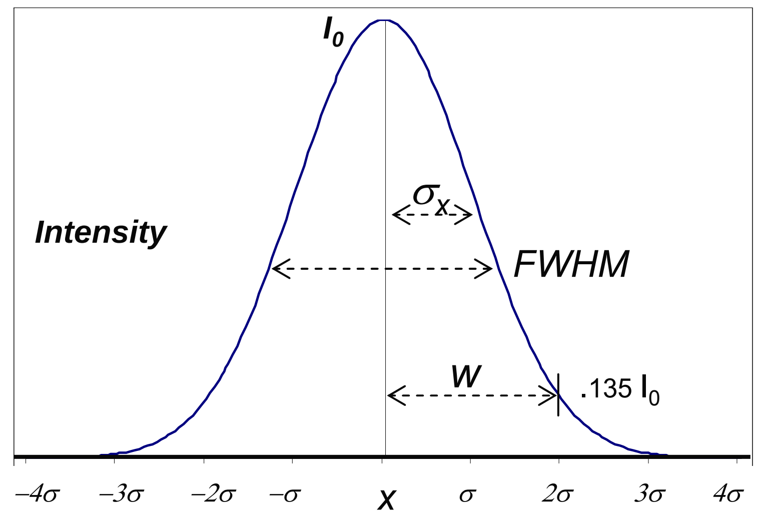
1/e² radius

$$w = 2\sigma_x$$

Full width at half max (FWHM)

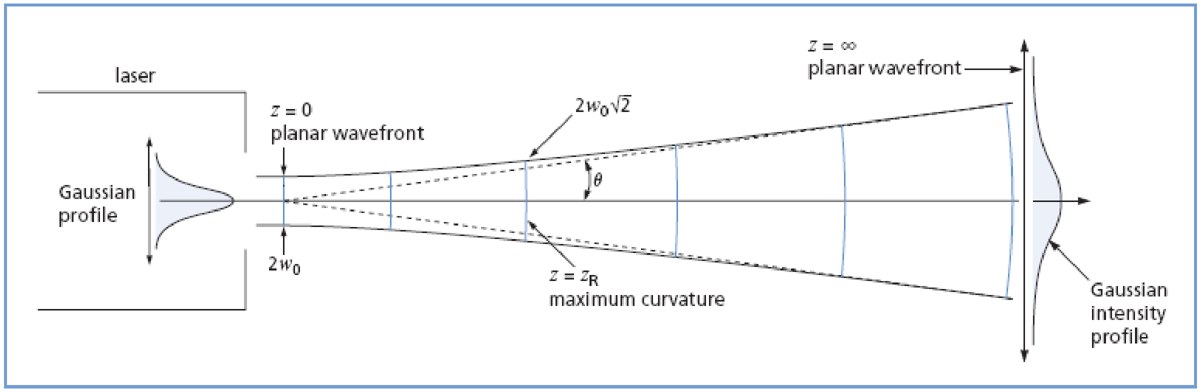
$$FWHM = 2\sqrt{2\ln 2}\sigma_x$$

$$FWHM = 2.355\sigma_x$$



$$\boxed{I(x,y) = \frac{2P}{\pi w^2} e^{-2\left(\frac{x}{w}\right)^2} e^{-2\left(\frac{y}{w}\right)^2}}$$

Gaussian Beam Propagation



Parabolic expansion of $1/e^2$ radius with z

$$\boxed{w^2 = w_0^2\left(1+\frac{z^2}{z_R^2}\right)}$$

Rayleigh length

$$z_R = \frac{\pi w_0^2}{\lambda}$$

At large z the divergence angle scales with λ/w_0

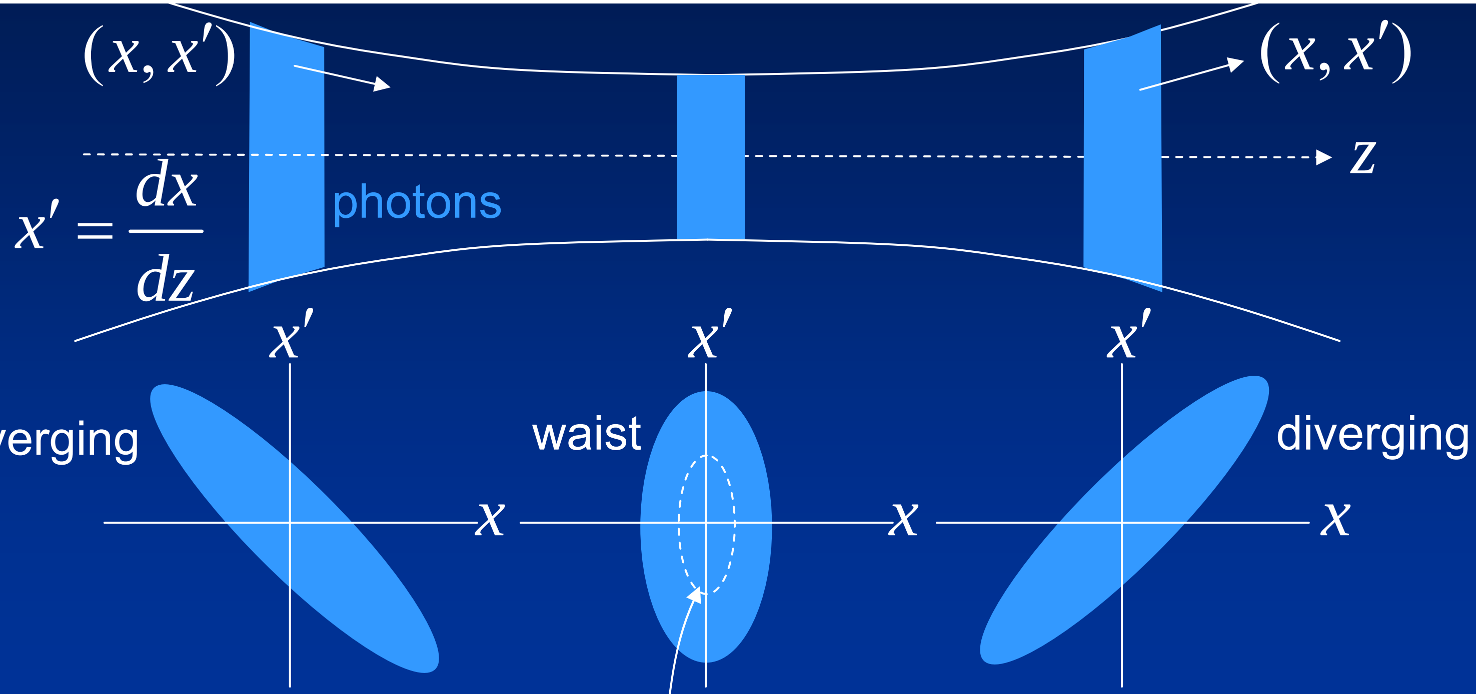
$$\theta = \frac{\lambda}{\pi w_0}$$

The product of the waist radius and converging angle of a diffraction limited beam is the wavelength divided by π. Focusing the beam to small spots requires large angles.

Diffraction limit

$$\boxed{w_0\theta = \frac{\lambda}{\pi}}$$

Laser Beam Emittance



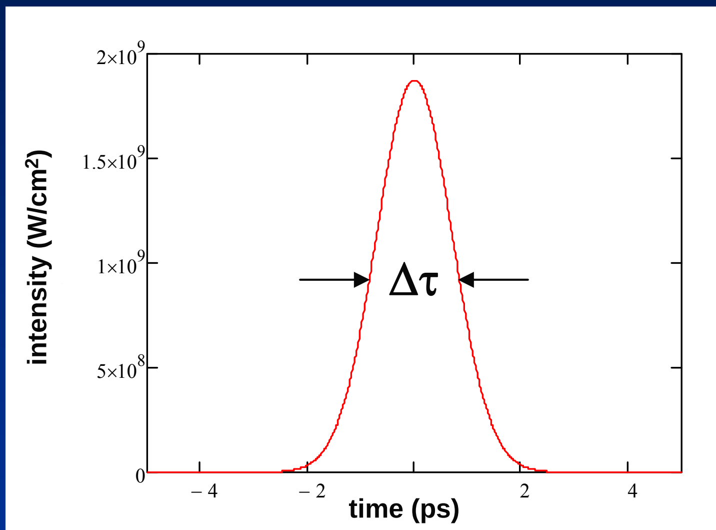
Light phase space area = π times σ_x (rms radius) times $\sigma_{x'}$ (rms angle)

$$A_{rms} = \pi \sigma_x \sigma_{x'} = \frac{\pi w_0 \theta}{4} = \frac{\lambda}{4}$$

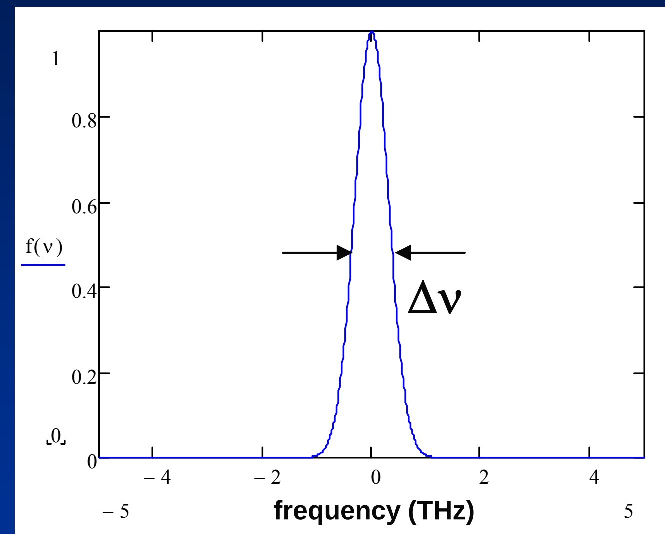
Photon beam emittance

$$\boxed{\sigma_x \sigma_{x'} = \frac{\lambda}{4\pi}}$$

Longitudinal Coherence



Fourier transform



If $\Delta\nu$ and $\Delta\tau$ are the full-width at half max (FWHM), the transform limit becomes

$$\boxed{\Delta\nu\Delta\tau = 0.44}$$

Gaussian pulse

$$I = I_0 e^{-\left(\frac{t}{\sigma_t}\right)^2} = I_0 e^{-4\ln 2\left(\frac{t}{\Delta\tau}\right)^2}$$

Coherence length

$$L_c = \frac{\lambda^2}{\Delta\lambda}$$

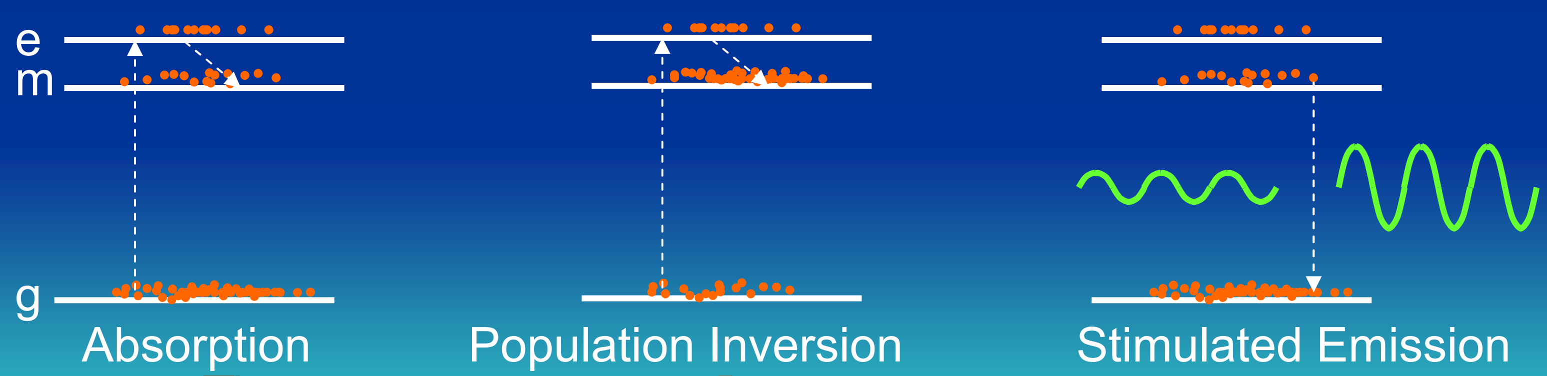
An optical pulse with length $\Delta\tau$ is fully coherent if its coherence length ≥ 2 c $\Delta\tau$

How a quantum laser works

An external source of energy excites electrons from the **g**round state to an **e**xcited state

Electrons from excited state decay to a **m**etastable energy level with long lifetime (transition from this level to the ground state is quantum mechanically forbidden) → population inversion

A co-propagating light beam stimulates emission of radiation → amplification of co-propagating light beam (**L**ight **A**mplification by **S**timulated **E**mission of **R**adiation)



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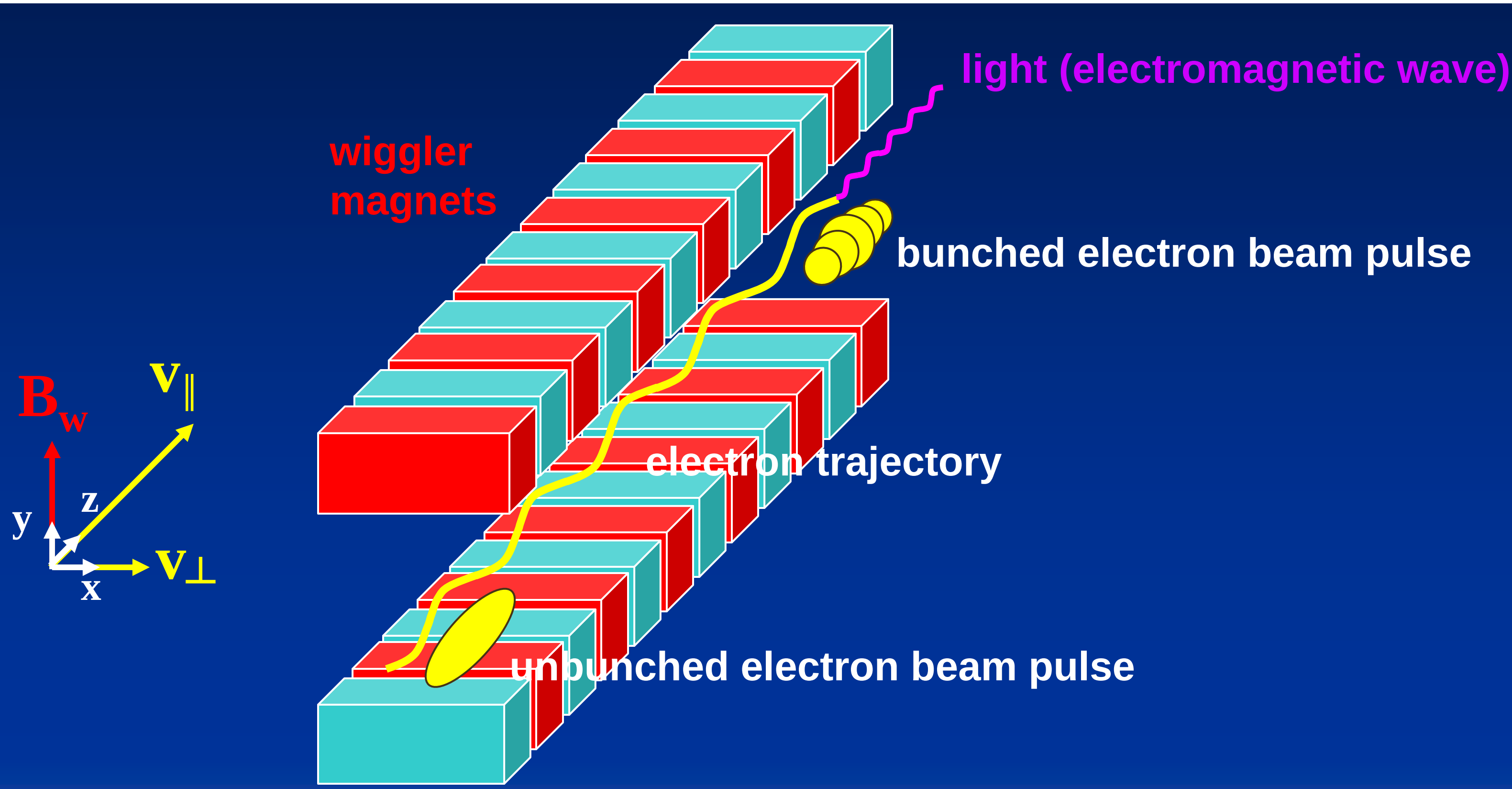
How an FEL works



Electrons in an FEL are not bound to atoms or molecules. The “free” electrons traverse a series of alternating magnets, called a “wiggler,” and radiate light at wavelengths depending on electrons’ energy, wiggler period and magnetic field.

How an FEL works (cont'd)

The wiggler induces transverse sinusoidal velocity in electron beam

Energy exchange occurs between the transverse electron current and transverse electric field of a co-propagating light beam

$$\boxed{\dot{W} = -e\upsilon_\perp \cdot E_s}$$

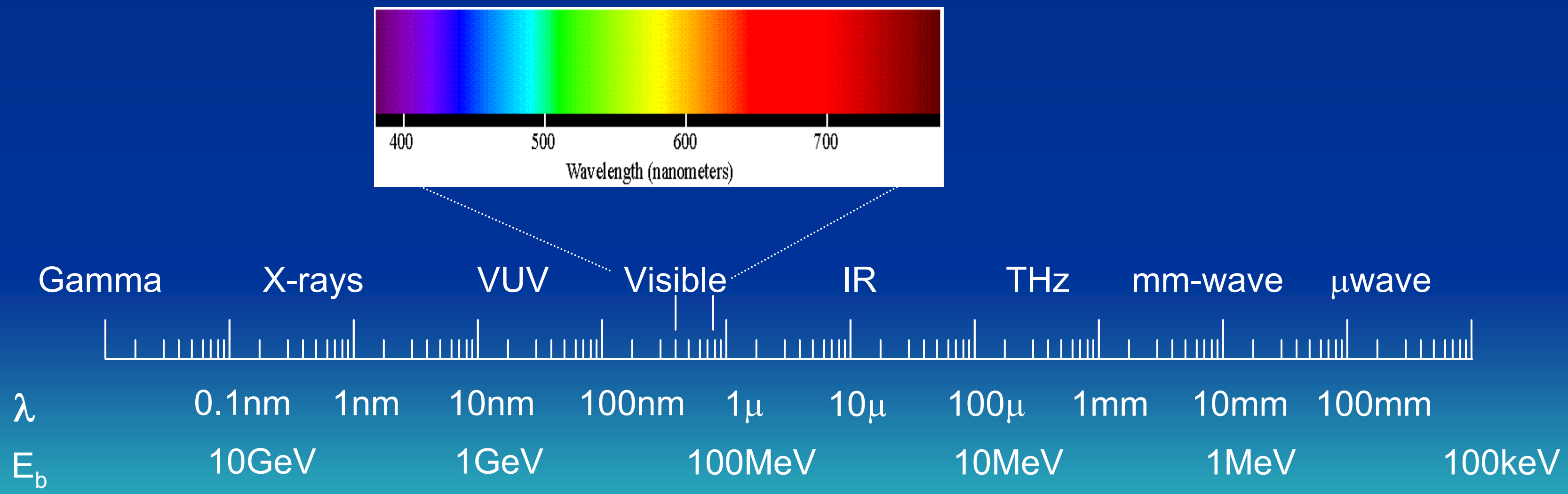
Depending on the phase of the light beam with the electrons' wiggling motion, some electrons gain energy while others lose energy $\rightarrow$ energy modulation $\rightarrow$ **bunching** of electrons along the axial direction into microbunches with period equal to an optical wavelength

Microbunched electron beams radiate coherently at higher power $\rightarrow$ **amplification** of the co-propagating light beam.

Note: The subscript $\perp$ denotes transverse and *s* stands for signal.

Basic features of FEL

- Wavelength tunable
- Diffraction limited optical beam
- Longitudinally and transversely coherent
- High power (GW peak, 100kW to MW average)
- Efficient (with energy recovery)



Wavelength Tunability

Select coarse wavelength by choosing the electron beam energy, wiggler period and wiggler magnetic field. Fine-tune wavelength by adjusting electron beam energy or wiggler magnetic field.

λ resonant wavelength

λ_w wiggler period

$$\boxed{\lambda = \frac{\lambda_w}{2\gamma^2}\left(1 + a_w^2\right)}$$

γ relativistic factor

a_w (also K_{rms}) rms wiggler parameter

$$a_w = \frac{eB_0}{\sqrt{2}k_w m_0 c} \approx 0.66 B_0(T)\lambda_w(cm)$$

For electrons (m_0c^2 = 0.511 MeV)

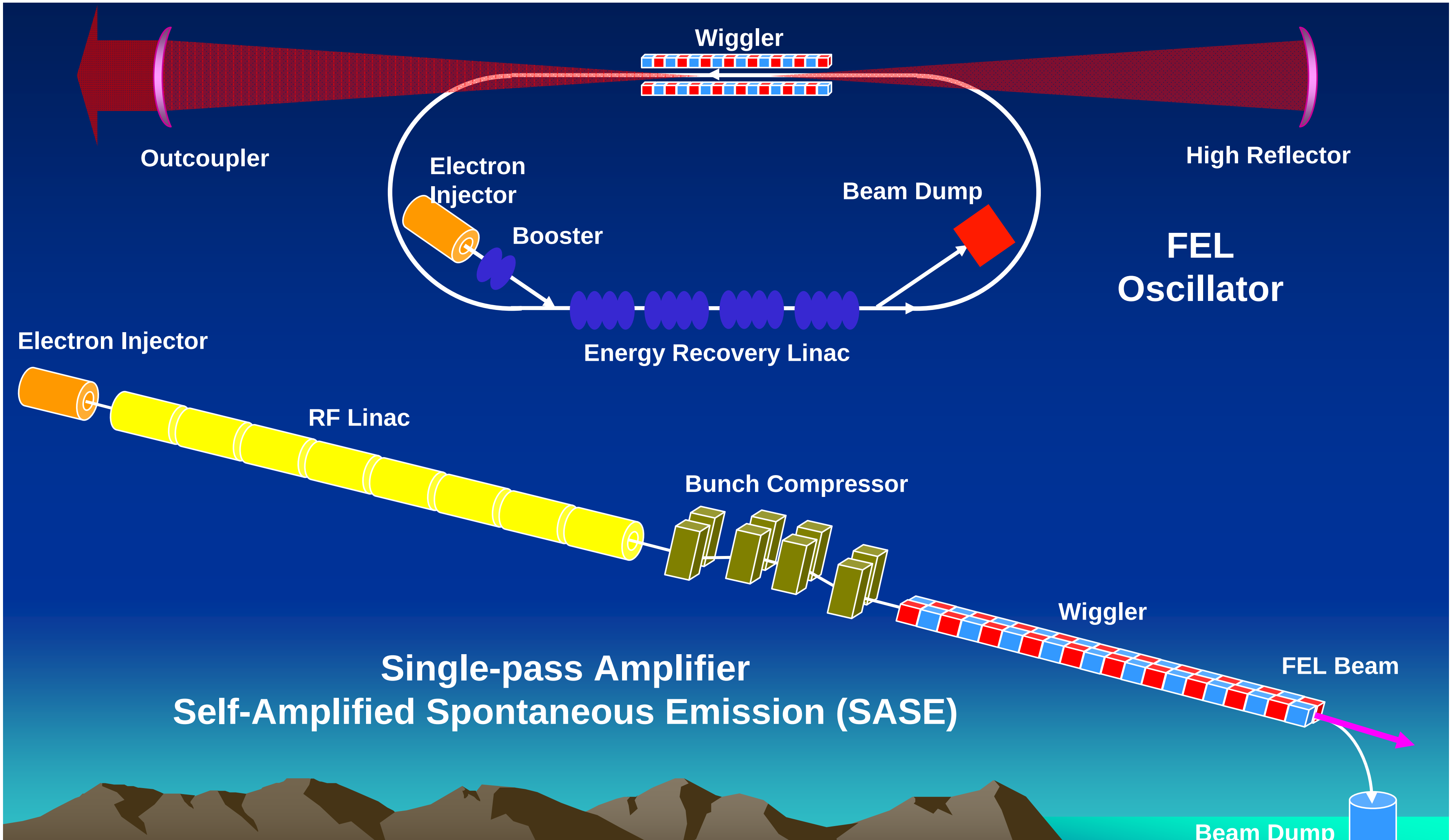
$$\gamma = \frac{T}{m_0c^2} + 1 \approx 2T(MeV)$$

Another convention uses peak parameter K

$$K = \frac{eB_0}{k_w m_0 c} = \sqrt{2}a_w$$

$$\lambda = \frac{\lambda_w}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

Radio-frequency Linac FEL



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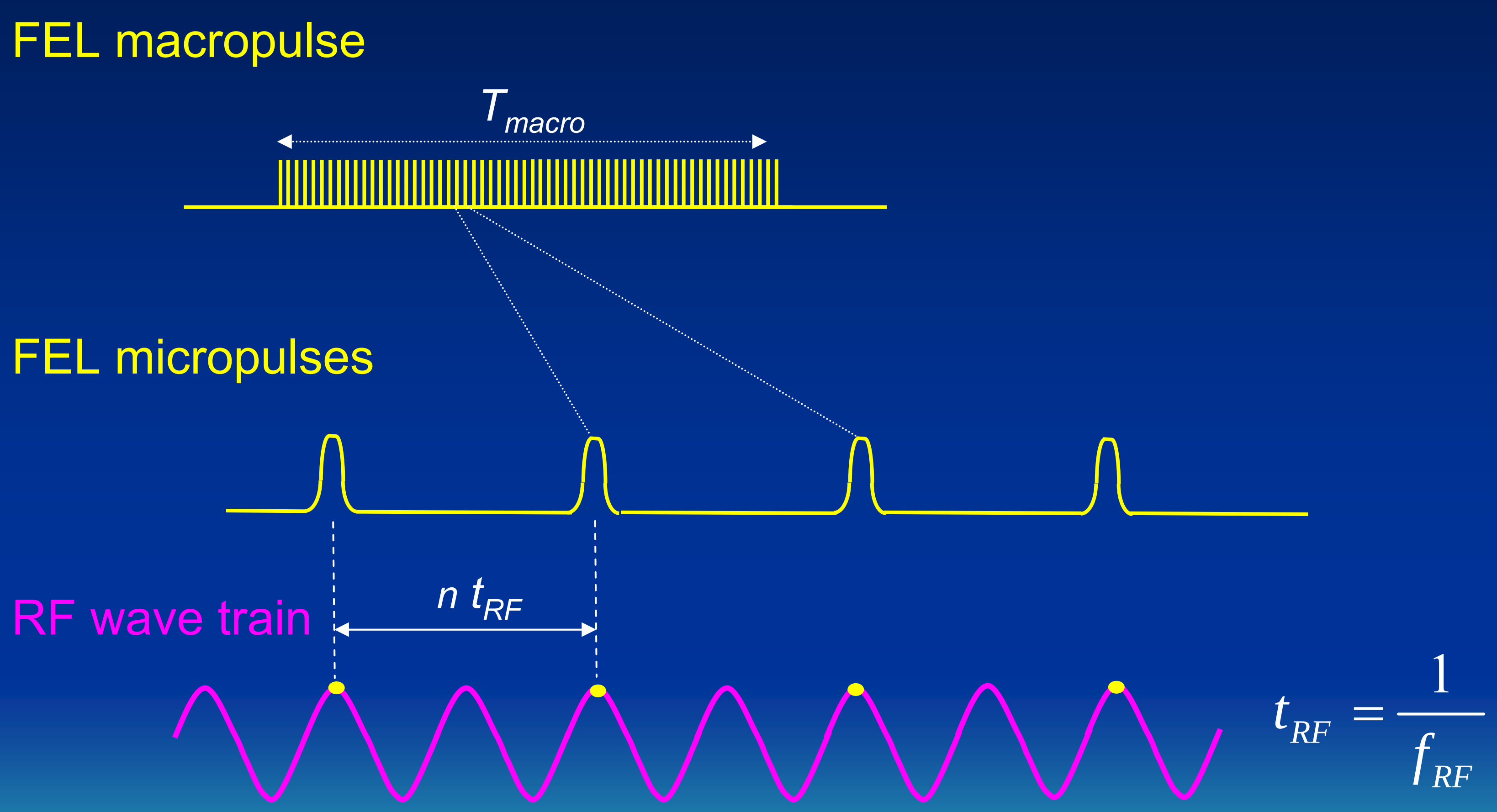
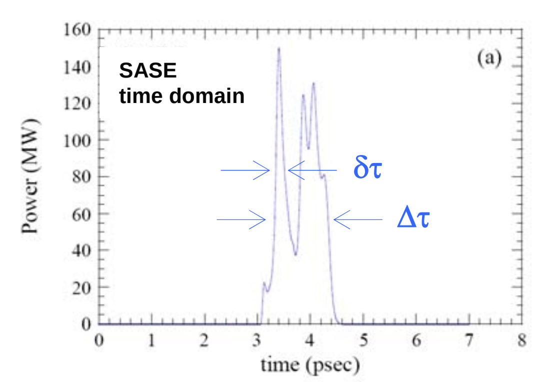
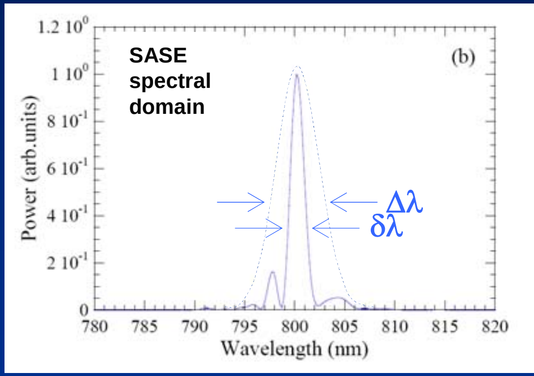
RF-Linac FEL Pulse Structure



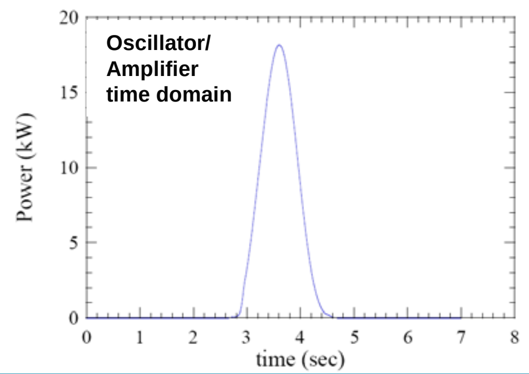
Temporal & Spectral Structures

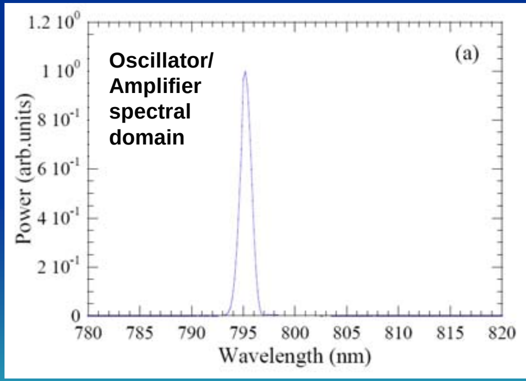
SASE FEL have spiky temporal and spectral features.





Unsaturated oscillator/amplifier FEL have smooth temporal and spectral profiles.





FEL optical beam properties

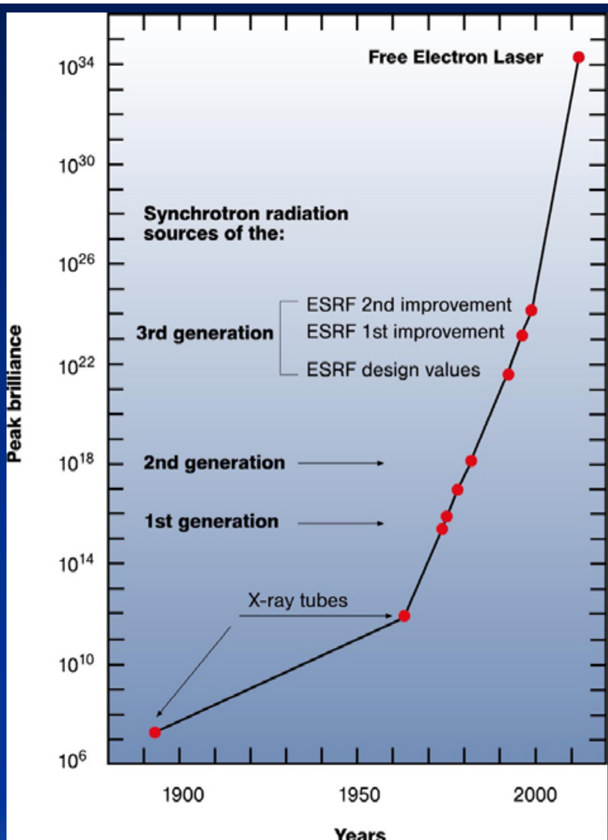
- Intensity
  $$I = \frac{2N_p h\nu}{\pi w_0^2 \Delta t}$$
  W/cm²

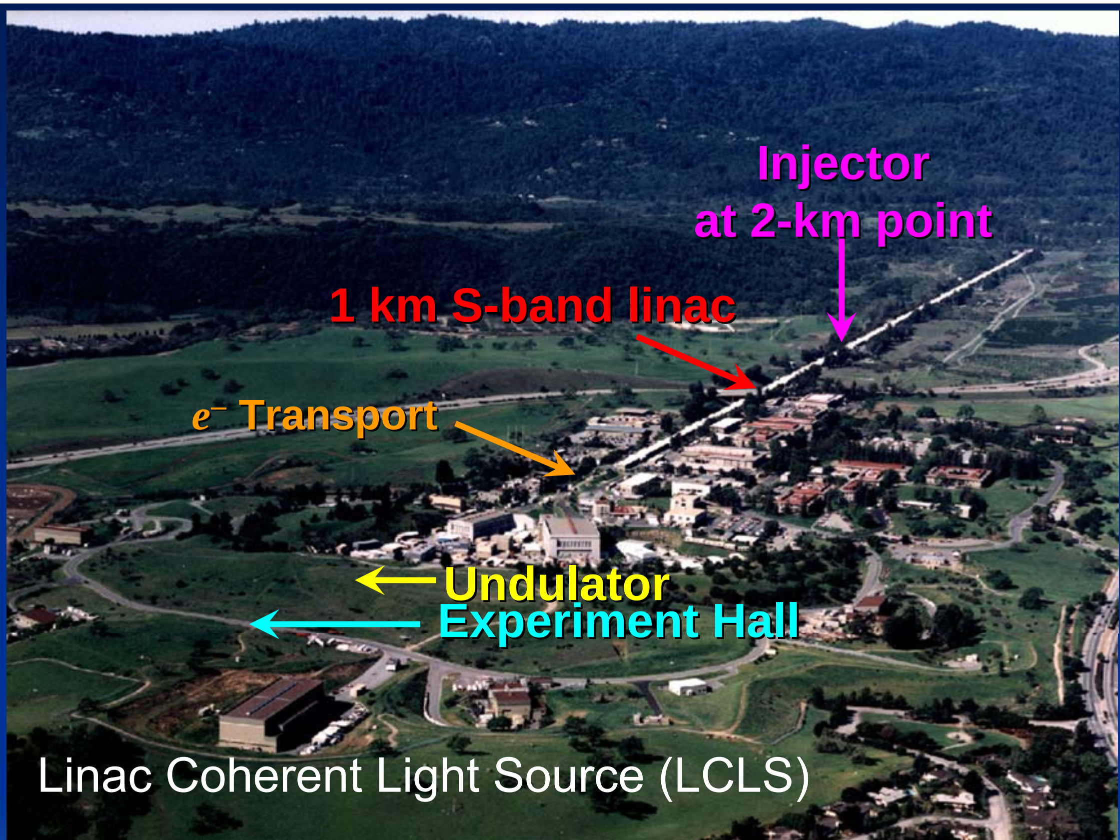
- Brightness
  $$B = \frac{N_p h\nu}{\pi^2 \varepsilon_x \varepsilon_y \Delta t}$$
  W/μm²

- Spectral bandwidth
  $$\frac{\Delta\omega}{\omega} \approx \frac{1}{N_w}$$

- Brilliance
  $$\mathfrak{B} = \frac{N_p}{\pi^2 \varepsilon_x \varepsilon_y \Delta t \frac{\Delta\omega}{\omega}}$$
  photons/(μm² s 0.1% BW)

4th Generation Light Source (4GLS)





Peak brilliance of linac-based 4th generation light sources (XFEL) is 8-10 orders of magnitude higher than that of 3rd generation light sources and >20 orders of magnitude above Bremsstrahlung sources.

Some examples of 4GLS

	LCLS	European XFEL	SCSS
Institution Location Country	SLAC Palo Alto, CA USA	DESY Hamburg Germany	Spring-8 Hyogo Japan
Wavelength X-ray energy	0.15 nm 8 keV	0.1 nm 12.4 keV	0.1 nm 12.4 keV
Beam energy	14.3 GeV	20 GeV	8 GeV
Linac type, frequency Length	NCRF, 2.856 GHz 1 km	SRF, 1.3 GHz 3.4 km	NCRF, 5.712 GHz 0.75 km
Gun type, frequency Cathode	NCRF, 2.856 GHz Cu photocathode	L-band RF gun Cs_2Te photocathode	Pulsed DC gun CeB_6 thermionic
Bunch charge	0.25 nC	1 nC	1 nC
Bunch length	75 fs	80 fs	250 fs
rms emittance	0.4 μm	1.4 μm	2 μm
Wiggler period a_w (K) Length	3 cm 2.62 (3.7) 55 m	3.56 cm 2.33 (3.3) 200 m	1.5 cm 1.3 (1.838) 50 m

Peak brilliance of 4GLS

Pulse energy ~ 1 mJ

Photon energy ~ 1 keV

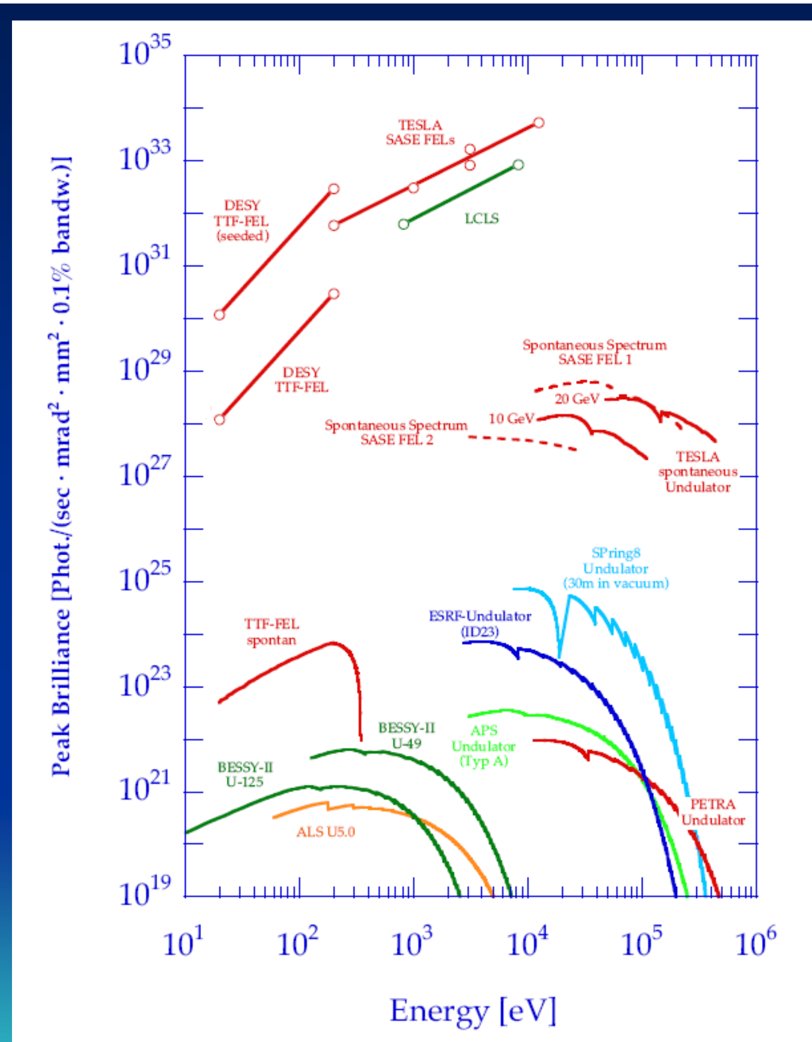
# of photons ~ 10^{13}

rms emittance ~ 10^{-4} μm

rms bunch length ~ 10^{-13} s

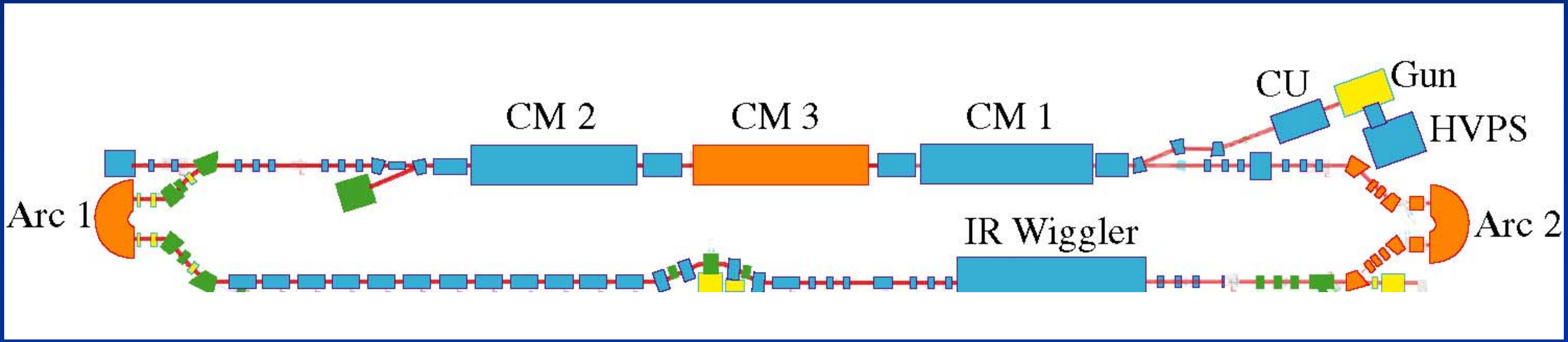
Energy spread ~ 0.01% BW

Brilliance ~ 10^{33} (s μm^2 0.1% BW)$^{-1}$



High-average-power FEL

- Ground-based FEL Program (Boeing/LANL, LLNL/TRW)
- Energy-recovery FEL (e.g. Jefferson Lab FEL)



Jefferson Lab FEL holds the world record in cw average power (14 kW).

Applications of FEL and 4GLS

FEL Features	*Wavelengths*	*Examples of applications*
• Ultrashort tunable pulses		
– Medicine	1-6 μm	Laser surgery
– Physics	XUV	Ultrafast spectroscopy
– Chemistry	XUV, UV	Chemical dynamics
– Biology	X-rays	Protein structures
• High peak power		
– High-density physics	X-rays	Warm dense matter
– Materials sciences	near-IR	Laser machining
• High average power		
– Directed energy	IR	Defense
– Space	near-IR	Power beaming
– Material processing	UV	Lithography

Chapter 2
Basics of Relativistic Dynamics

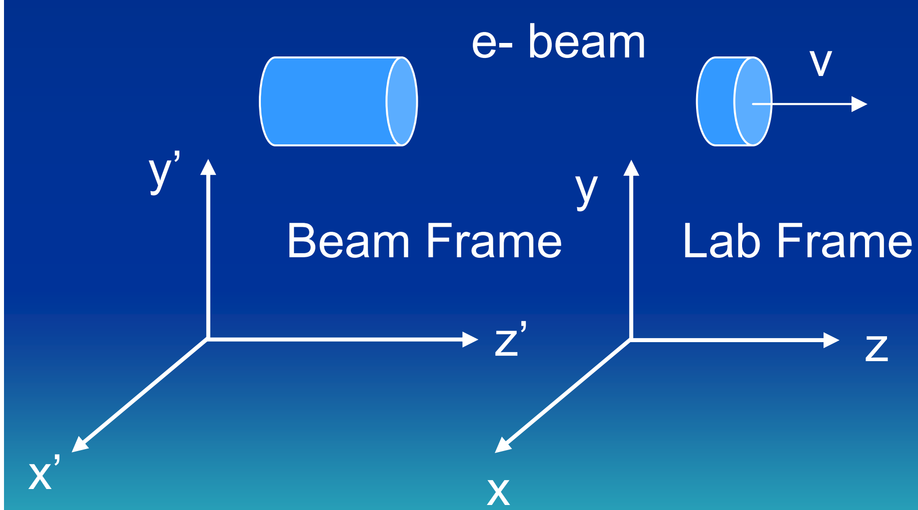
Basics of Relativistic Dynamics

- Special relativity
- Lorentz transformation
- Relativistic Doppler shifts
- Wavelength dependence on angle
- Relativistic velocity, momentum & energy
- Lorentz force law
- Curvilinear coordinate system
- Linear beam dynamics
- Emittance
- Emittance & energy spread requirements

Special Relativity

1. All inertial frames are completely equivalent with regard to physical phenomena

2. The speed of light in vacuum is the same for all observers in inertial frames of reference.



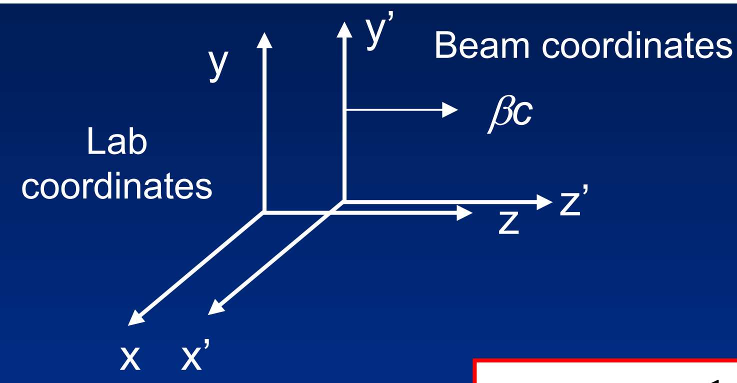
Lorentz Transformation

$$x' = x$$

$$y' = y$$

$$z' = \gamma\left(z - \beta ct\right)$$

$$ct' = \gamma\left(ct - \beta z\right)$$



Lorentz factor

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Velocity relative to c

$$\beta = \frac{v}{c}$$

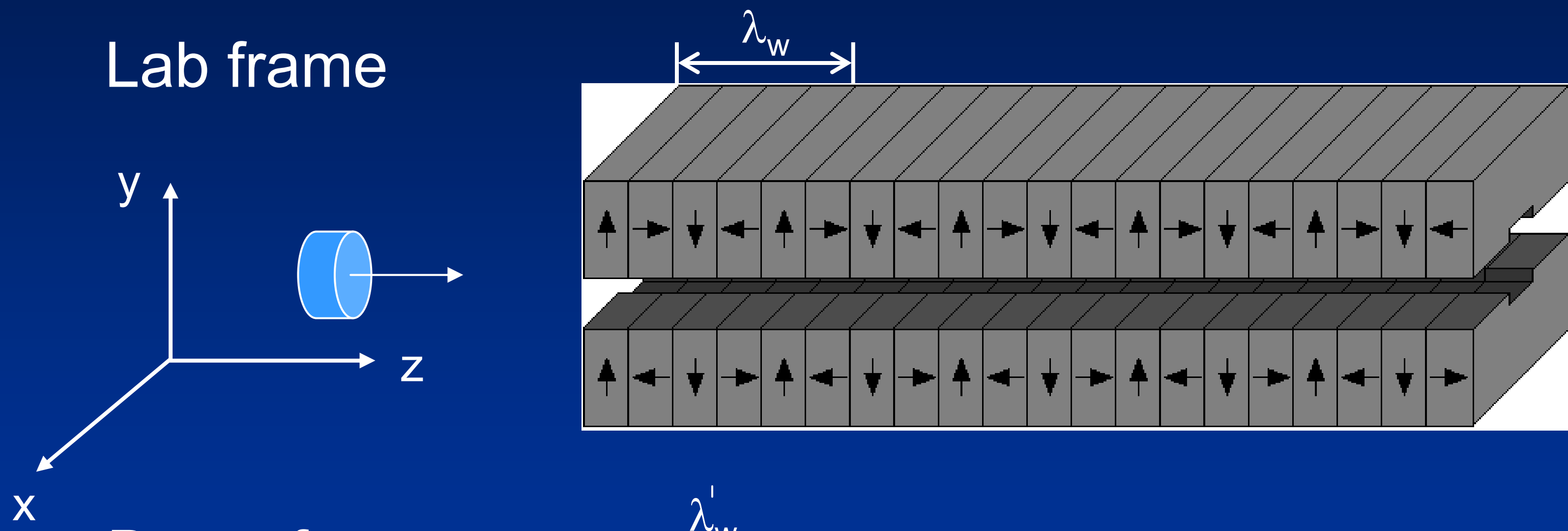
Transverse dimensions are unchanged.

Lengths of moving objects along direction of motion appear to be contracted in the Lab frame by a factor γ (Lorentz-FitzGerald contraction)

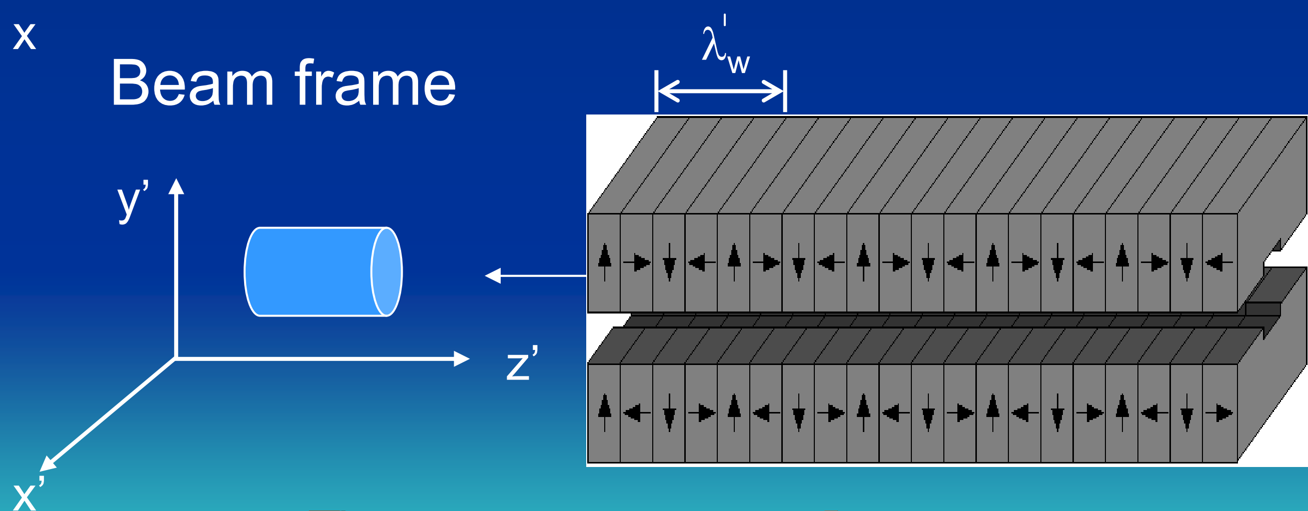
Clocks in the moving objects run slower by γ as observed in the Lab frame (time dilation).

Wiggler period contracts in beam frame

Lab frame



Beam frame



Wiggler period in beam frame

$$\lambda_w' = \frac{\lambda_w}{\gamma}$$

Lorentz Transformation of Fields

Electric field

$$E_x' = \gamma\left(E_x + \upsilon_z B_y\right)$$

$$E_y' = \gamma\left(E_y - \upsilon_z B_x\right)$$

$$E_z' = E_z$$

Magnetic field

$$B_x' = \gamma\left(B_x - \frac{\upsilon_z}{c^2} E_y\right)$$

$$B_y' = \gamma\left(B_y + \frac{\upsilon_z}{c^2} E_x\right)$$

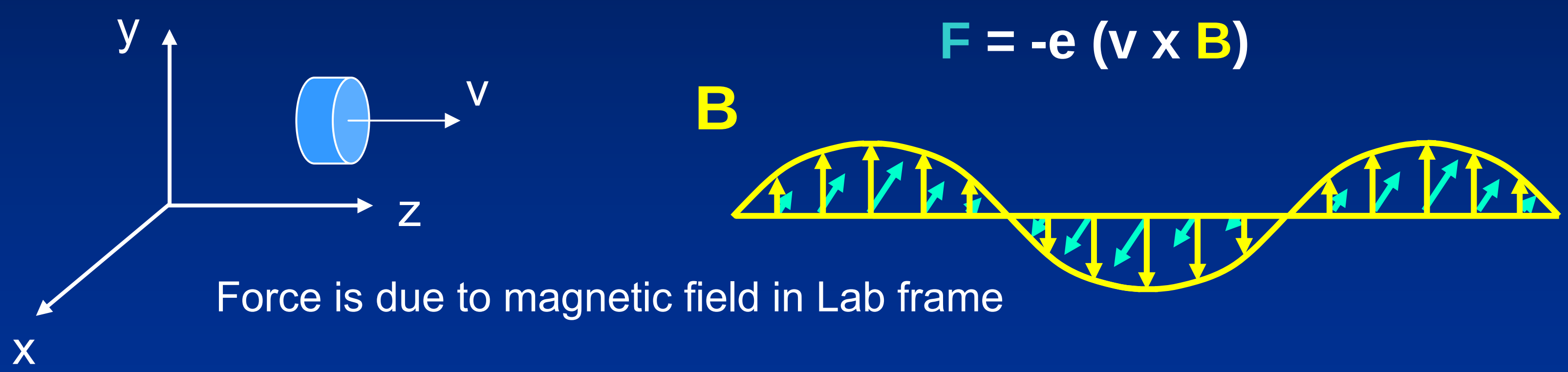
$$B_z' = B_z$$

Transverse electric and magnetic fields are different in the beam frame. Pure electric (and magnetic) fields in the Lab frame transform into mixed electric and magnetic fields in the beam frame. Longitudinal (along the direction of motion) electric and magnetic fields remain the same.

Electromagnetic Field Transformation

Lab frame

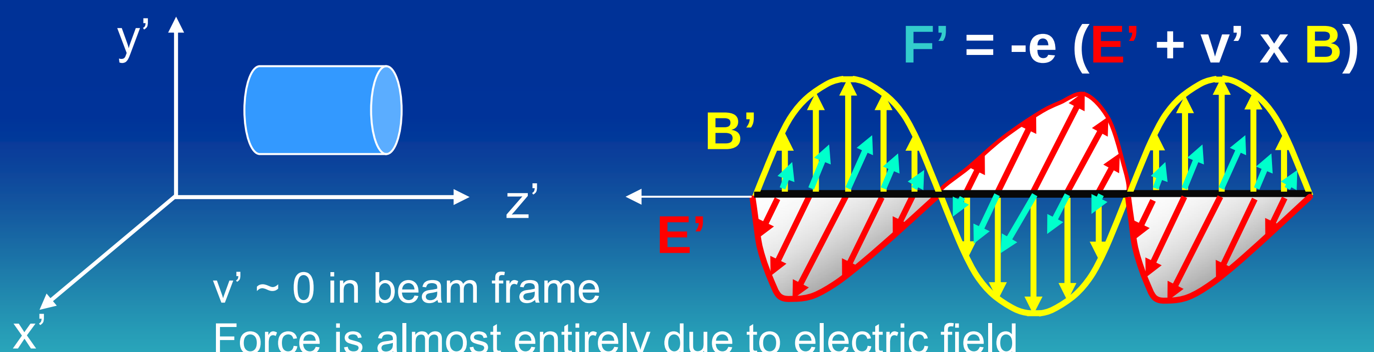
Wiggler magnetic field deflects electrons in x direction



Force is due to magnetic field in Lab frame

Beam frame

Electromagnetic field deflects electrons in x' direction

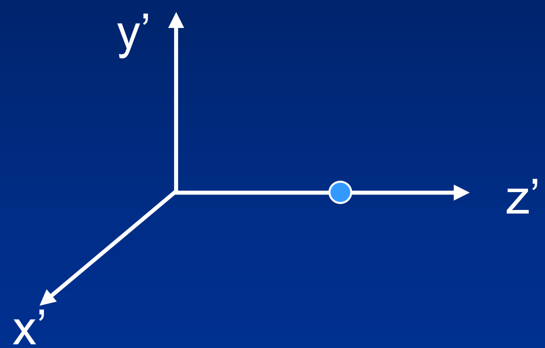


v' ~ 0 in beam frame

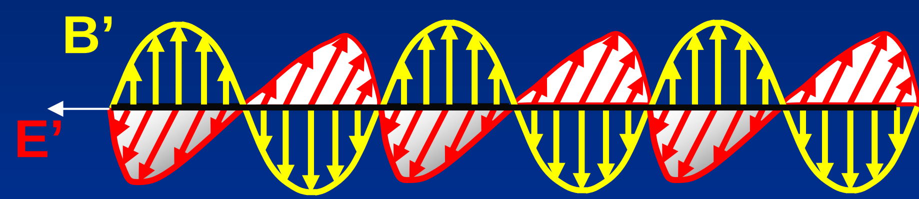
Force is almost entirely due to electric field

Radiation in Beam Frame

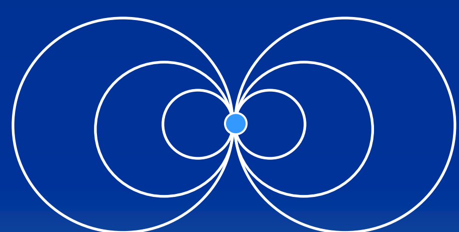
Beam frame



Wiggler electromagnetic wave behaves like virtual photons impinging on the electrons



View from the top



Real photons are scattered off the electrons. They can also be seen in the beam frame as circular waves radiated from the electrons at frequency ν'

$$\nu' \approx \frac{c}{\lambda_w'} = \frac{\gamma c}{\lambda_w}$$

Lorentz contraction causes ν' to be increased by a factor of γ compared to Lab frame

Useful Relativistic Relations

Exact relations

$$\gamma^2 = \frac{1}{1-\beta^2}$$

$$\beta^2 = 1 - \frac{1}{\gamma^2}$$

$$\beta^2\gamma^2 = \gamma^2 - 1$$

$$\frac{1}{\beta^2\gamma^2} = \frac{1}{\beta^2} - 1$$

Approximations for $\beta \sim 1$

$$\beta \approx 1 - \frac{1}{2\gamma^2}$$

$$1 - \beta \approx \frac{1}{2\gamma^2}$$

$$\frac{1}{\beta} \approx 1 + \frac{1}{2\gamma^2}$$

$$\frac{1}{\beta} - 1 \approx \frac{1}{2\gamma^2}$$

Lorentz Transformation of Frequency and Angle

Relativistic Doppler shift depends on Lab frame observation angle

$$\nu' = \gamma\left(1-\beta\cos\theta\right)\nu$$

$$\nu = \frac{\nu'}{\gamma\left(1-\beta\cos\theta\right)}$$

Backward ($\theta = \pi$)

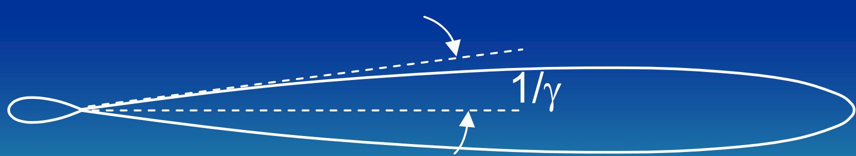
$$\boxed{\nu = \frac{\nu'}{2\gamma}}$$

Forward ($\theta = 0$)

Use approximation $1-\beta \approx \dfrac{1}{2\gamma^2}$

Relativistic Doppler shift in the forward direction $\boxed{\nu = 2\gamma\nu'}$

For γ >>1 Lorentz transformation yields $1/\gamma$ emission angle

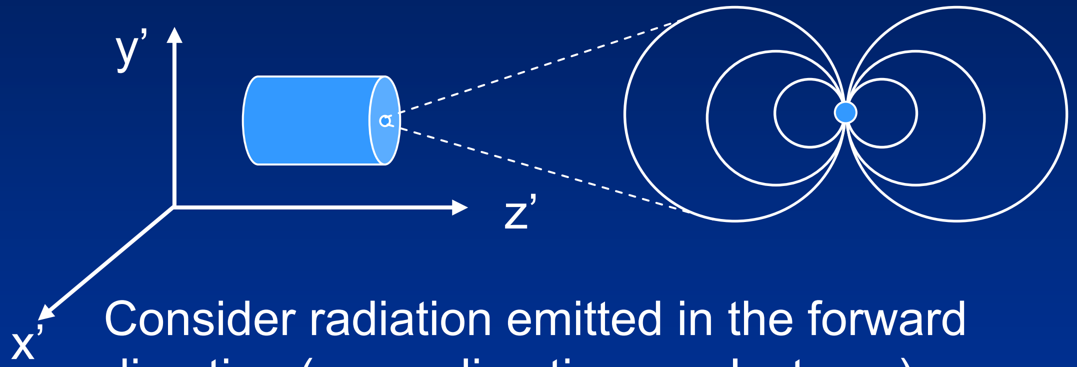


$$1-\cos\theta = \gamma^2\left(1-\beta\cos\theta\right)^2\left(1-\cos\theta'\right)$$

For small angles $\theta = \dfrac{\sqrt{1-\cos\theta'}}{\gamma}$

Longitudinal Doppler Shift (Forward)

Beam frame



Doppler effect causes 2γ up-shift in frequency and narrowing of emission angle

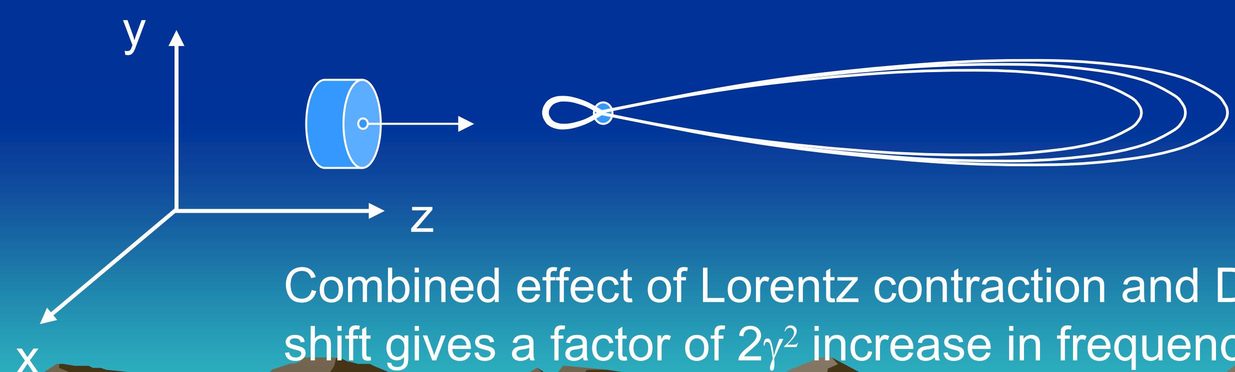
$$\nu = 2\gamma\nu'$$

Consider radiation emitted in the forward direction (same direction as electrons)

$$\nu = 2\gamma\nu' = 2\gamma\left(\frac{\gamma c}{\lambda_w}\right)$$

$$\nu = \frac{2\gamma^2 c}{\lambda_w}$$

Lab frame

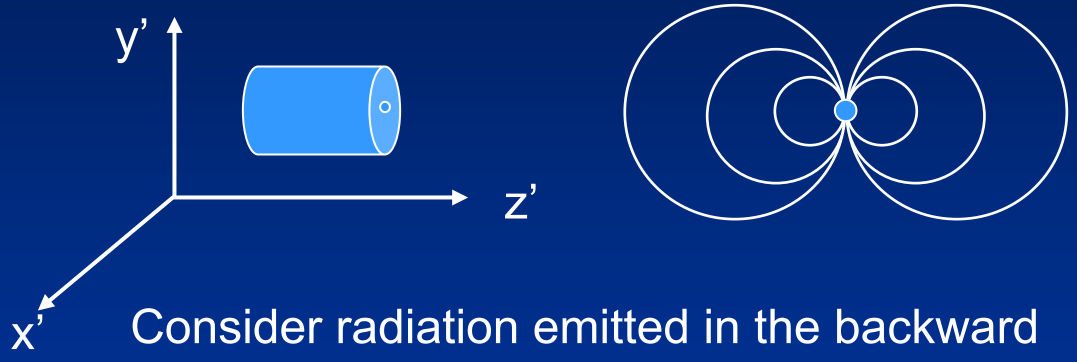


Combined effect of Lorentz contraction and Doppler shift gives a factor of $2\gamma^2$ increase in frequency

$$\boxed{\lambda = \frac{\lambda_w}{2\gamma^2}}$$

Longitudinal Doppler Shift (Backward)

Beam frame



Doppler effect causes 2γ down-shift in frequency

$$\nu = \frac{\nu'}{2\gamma}$$

Consider radiation emitted in the backward direction (opposite to beam direction)

Lab frame

$$\nu = \frac{\nu'}{2\gamma} = \frac{1}{2\gamma}\left(\frac{\gamma c}{\lambda_w}\right) = \frac{c}{2\lambda_w}$$

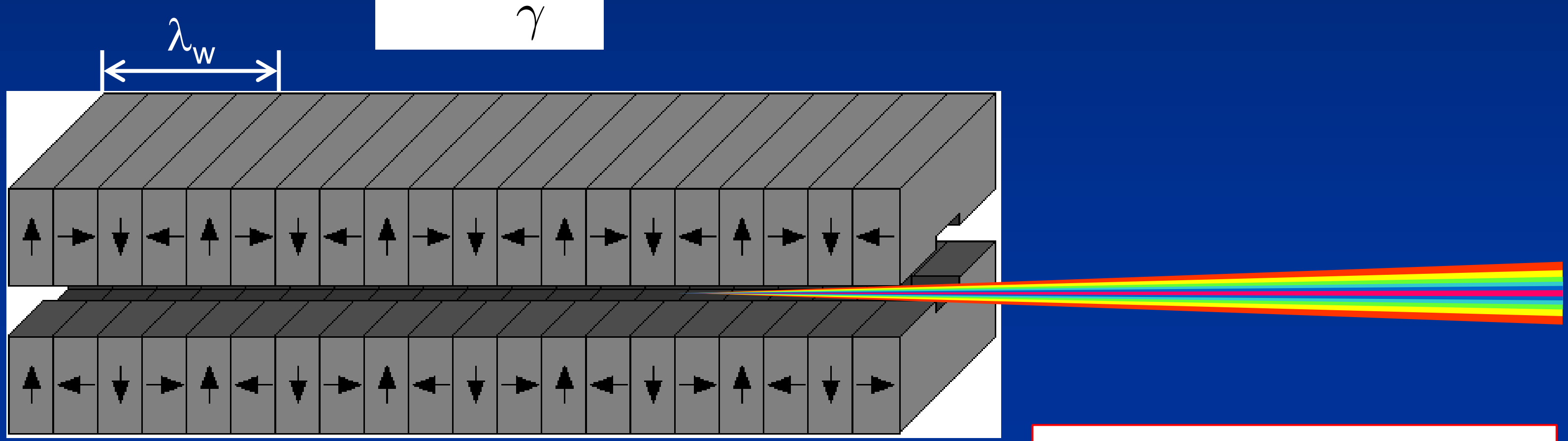
Lorentz contraction is negated by Doppler shift. Frequency is reduced by a factor of 2.

$$\lambda = 2\lambda_w$$

Wavelength Dependence on Angle

The wavelength of wiggler (undulator) radiation depends on emission angle. Shortest wavelengths are radiated in the forward direction ($\theta = 0$). Radiation at larger angles have longer wavelengths. The opening half angle of wiggler radiation, θ is given by

$$\theta = \frac{\sqrt{2}a_w}{\gamma}$$



$$\lambda = \lambda_w\left(1+\beta\right)$$

Backward wave

$$\lambda \approx \frac{\lambda_w}{2\gamma^2}\left(1+\gamma^2\theta^2\right)$$

Forward wave

Relativistic Energy & Momentum

Total energy $$E = T + m_0c^2 = mc^2 = \gamma m_0 c^2$$

Kinetic energy $$T = (m - m_0)c^2 = m_0c^2(\gamma - 1)$$

Momentum $$p = m\upsilon = \beta\gamma m_0 c$$

Energy is in unit of MeV or GeV.

Momentum is in unit of MeV/c or GeV/c

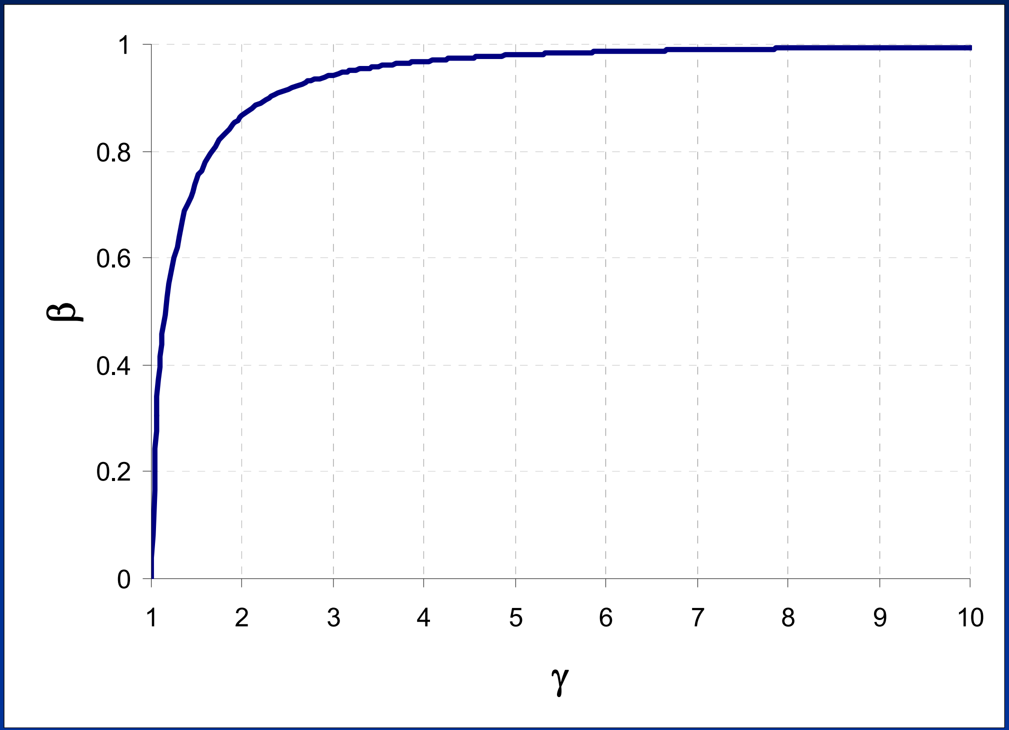
Multiply by *c* and square $$(cp)^2 = \beta^2\gamma^2(m_0c^2)^2 = (\gamma^2 - 1)(m_0c^2)^2$$

Energy right triangle $$E^2 = (cp)^2 + (m_0c^2)^2$$

Parameter Variation Table

	$\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$\frac{d\gamma}{\gamma}$
$\frac{d\beta}{\beta} =$	1	$\frac{1}{\gamma^2}$	$\frac{1}{\beta^2\gamma^2}$
$\frac{dp}{p} =$	γ^2	1	$\frac{1}{\beta^2}$
$\frac{d\gamma}{\gamma} =$	$\beta^2\gamma^2$	β^2	1

Relative velocity differences become smaller at high energy

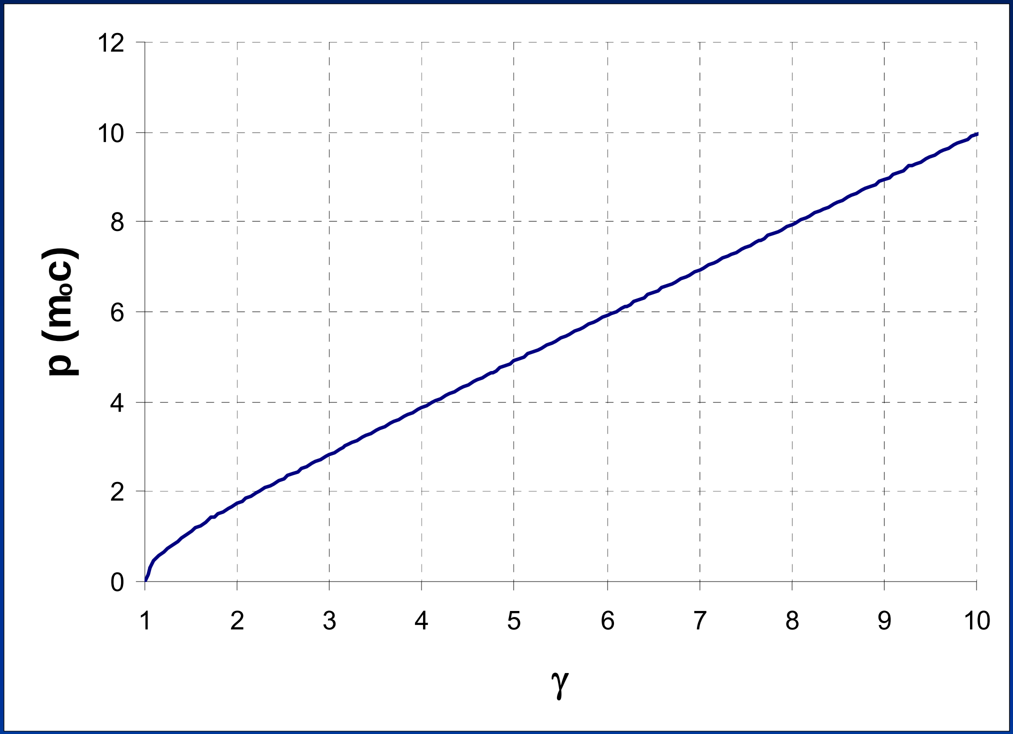


$$\frac{d\beta}{\beta} = \frac{1}{\beta^2\gamma^2}\frac{d\gamma}{\gamma}$$

Most electron accelerators are speed-of-light (β=1) machines

At large γ, it becomes very hard to perform ballistic bunch compression because all electrons travel nearly at the speed of light.

Relative momentum change is same as energy spread at high energy



$$\frac{dp}{p} = \frac{1}{\beta^2}\frac{d\gamma}{\gamma}$$

Bunch compression via momentum spread can be done at any energy

Given sufficient energy spread and dispersive elements such as magnetic chicanes, electron bunches can be compressed to ultrashort pulses.

Lorentz Force Law

$$\boxed{\mathbf{F} = -e\left[\mathbf{E} + \left(\mathbf{v} \times \mathbf{B}\right)\right]}$$

In MKS units, e = 1.6 x 10^{-19} coulomb, electric field is in volts/m and magnetic field is in tesla.

Electric force acts on electrons along their direction of motion and thus changes the electrons' kinetic energy.

$$\Delta T = \int \mathbf{F} \cdot d\mathbf{s} = -\int e\mathbf{E} \cdot d\mathbf{s}$$

Magnetic force is perpendicular to direction of motion and does not change the electrons' kinetic energy. Magnetic field can be used to change momentum, i.e. bend electron beams.

$$\Delta \vec{p} = \int \mathbf{F} dt = -e\int \left(\vec{\upsilon} \times \mathbf{B}\right) dt$$

Bending Relativistic Beams

Bend angle and radius

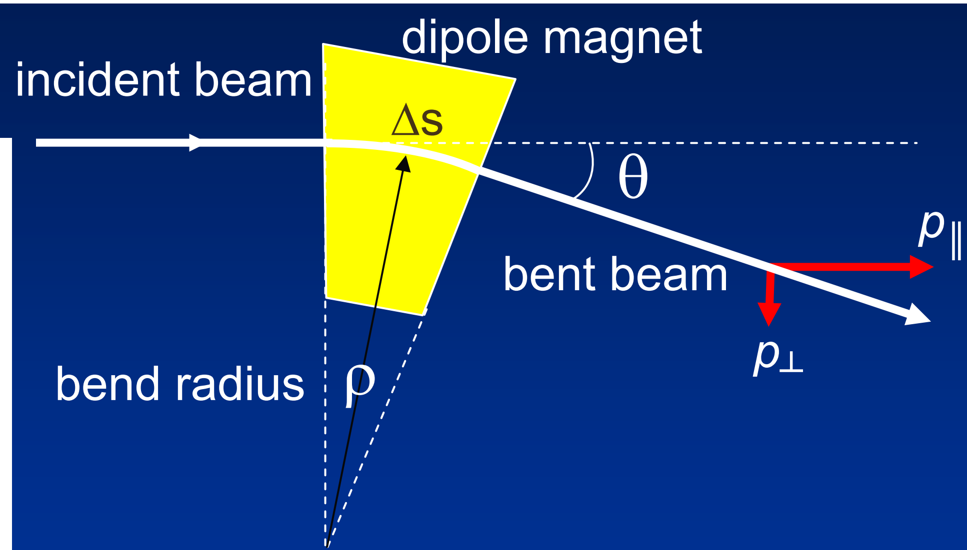
$$\tan\theta = \frac{p_\perp}{p_\parallel}$$

$$p_\perp = F\Delta t = -evB\Delta t = -eB\Delta s$$

$$p_\parallel = \beta\gamma m_0 c = \frac{E_b}{c}$$

$$\tan\theta = \frac{\Delta s}{\rho} = \frac{-ecB\Delta s}{E_b}$$

$$\frac{1}{\rho} = \left|\frac{ecB}{E_b}\right|$$

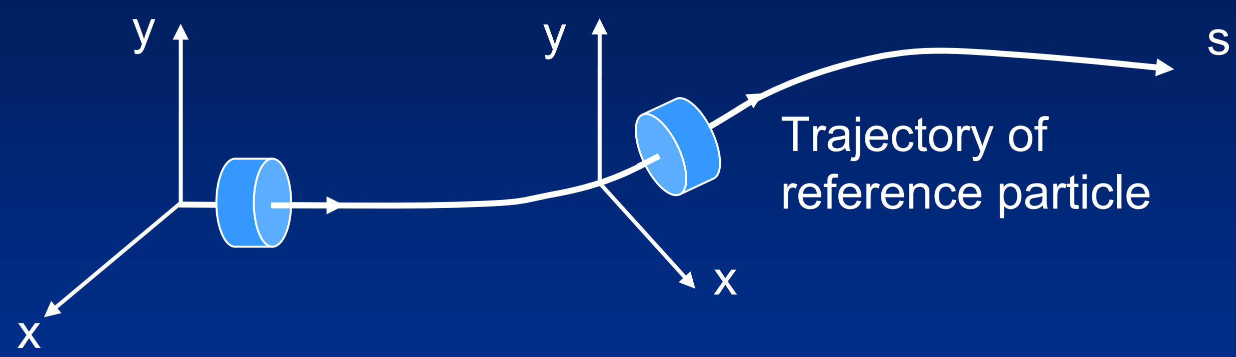


$$\frac{1}{\rho}\left(m^{-1}\right) = 299.8\frac{B(T)}{E_b\left(MeV\right)}$$

Magnetic rigidity

$$\boxed{B\rho\left(T-m\right) = \frac{1}{299.8}E_b\left(MeV\right)}$$

Curvilinear Coordinate



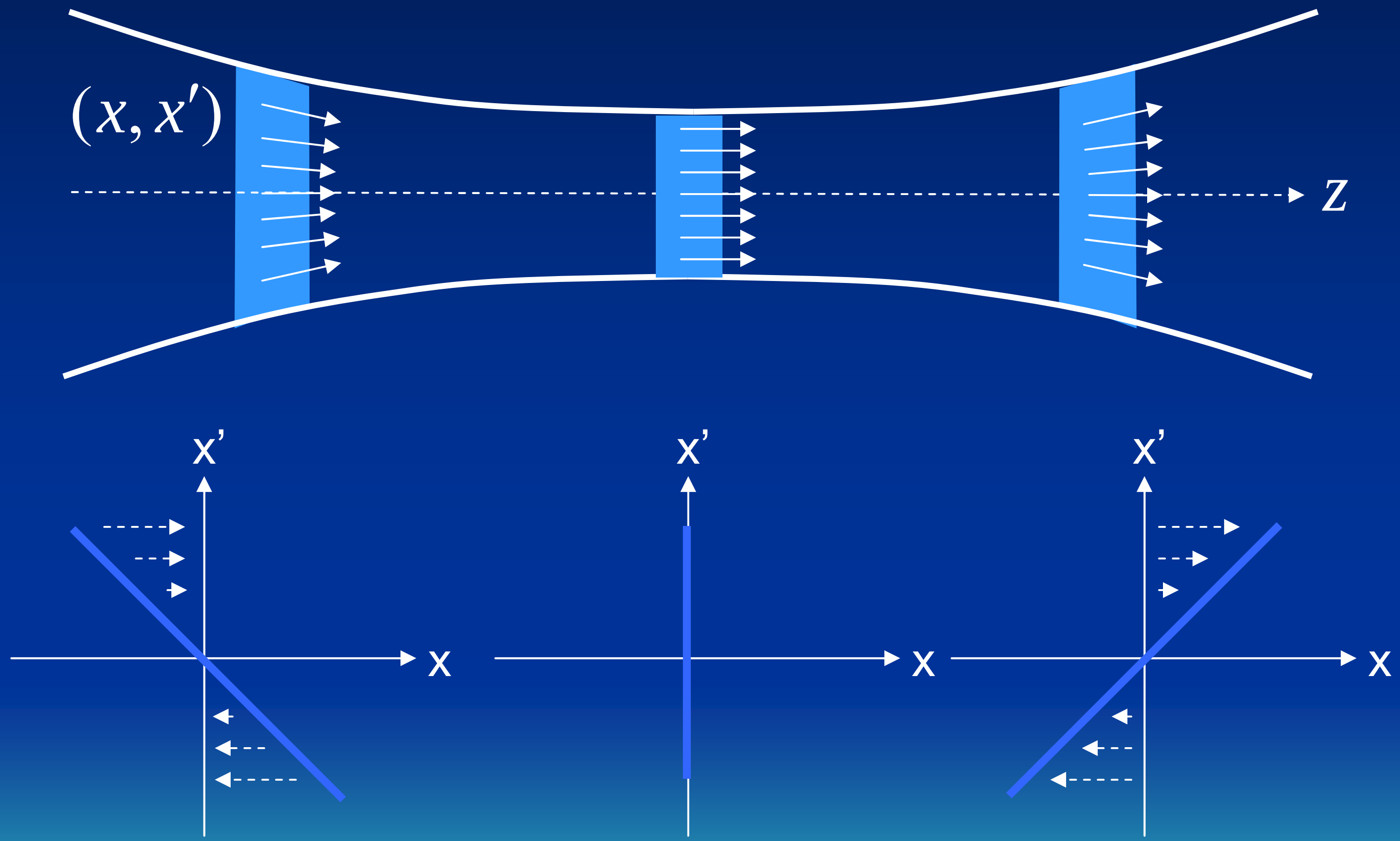
Electrons travel in the s direction. Use (x, y, s) coordinate system to follow the reference electron, an ideal particle at the beam center with a curvilinear trajectory. The reference particle trajectory takes into account only pure dipole fields along the beam line. The x and y of the reference trajectory are thus affected only by the placement and strength of the dipole magnets**.**

For other electrons, define x' and y' as the slopes of x and y with respect to s

$$x' = \frac{dx}{ds} \qquad y' = \frac{dy}{ds}$$

Paraxial Rays & Trace Space

Paraxial ray approximation deals with non-crossing trajectories near the axis.



In a drift space, converging beams come to a waist and then diverge

Lorentz Forces

Lorentz force in x

$$\gamma m_0 \frac{d^2 x}{dt^2} = -e\upsilon B_y \approx -ecB_y$$

Slope of x with respect to s

$$x' = \frac{dx}{ds} \approx \frac{1}{c}\frac{dx}{dt}$$

Curvature of x with respect to s

$$x'' = \frac{d^2 x}{ds^2} \approx \frac{1}{c^2}\frac{d^2 x}{dt^2}$$

$$x'' = \frac{-ecB_y}{\gamma m_0 c^2} \approx \frac{-eB_y}{p}$$

Lorentz force in y

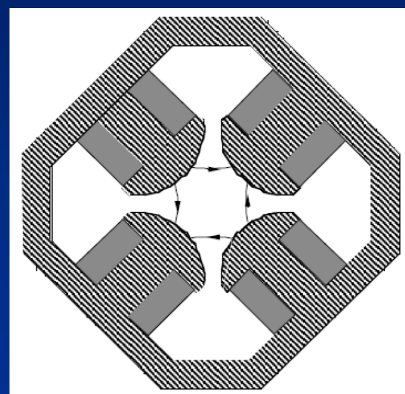
$$\gamma m_0 \frac{d^2 y}{dt^2} = e\upsilon B_x \approx ecB_x$$

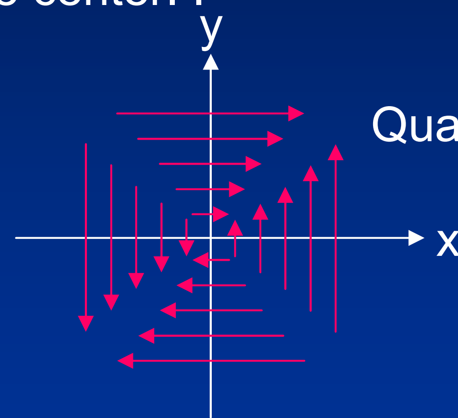
$$y'' = \frac{eB_x}{\gamma m_0 c} \approx \frac{eB_x}{p}$$

Quadrupole Lens

A quadrupole is a focusing element in one plane (e.g., x) and defocusing in the other plane (e.g., y). Its magnetic field, and thus the focusing force, increases linearly with distance from the center. .

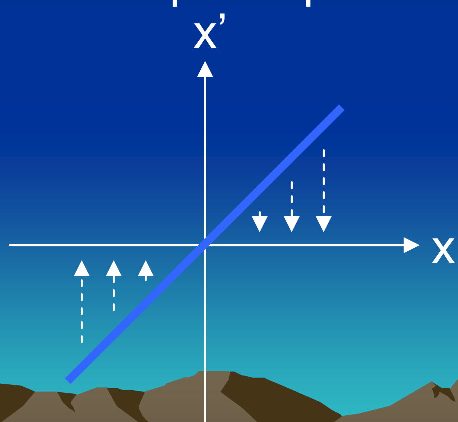
Quadrupole



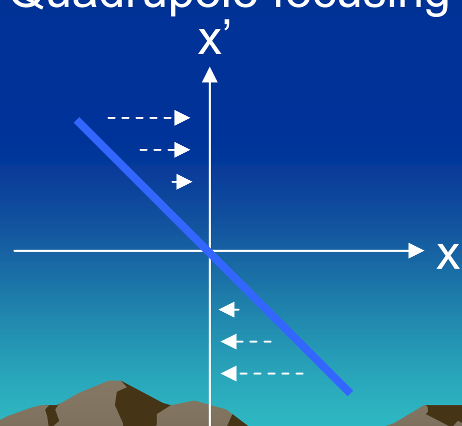


Quadrupole field

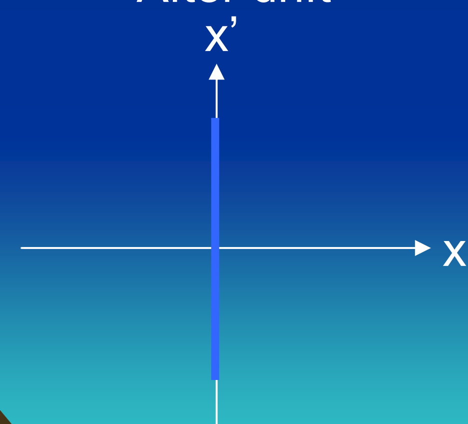
Before quadrupole



Quadrupole focusing



After drift



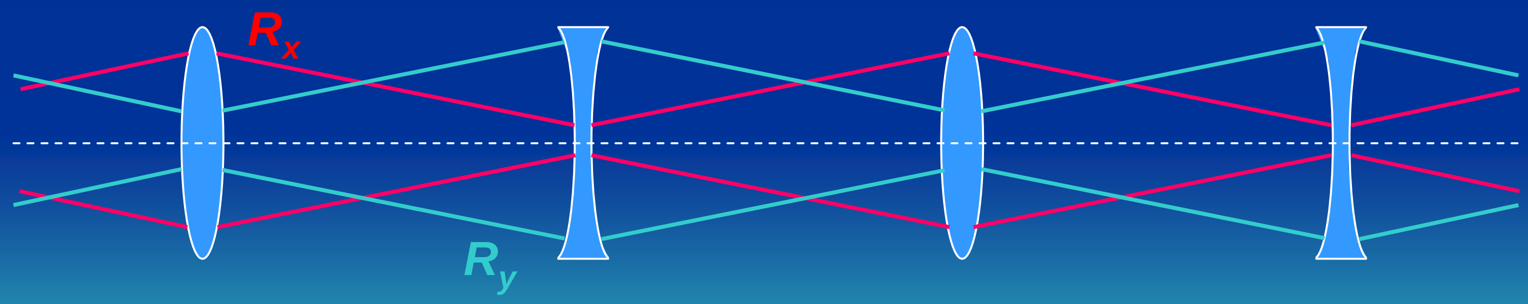
Linear Beam Dynamics

Linear beam dynamics is valid if the restoring forces in x and y are linear. Quadrupoles are linear focusing (and defocusing) elements since the restoring forces are linear with distance from the center.

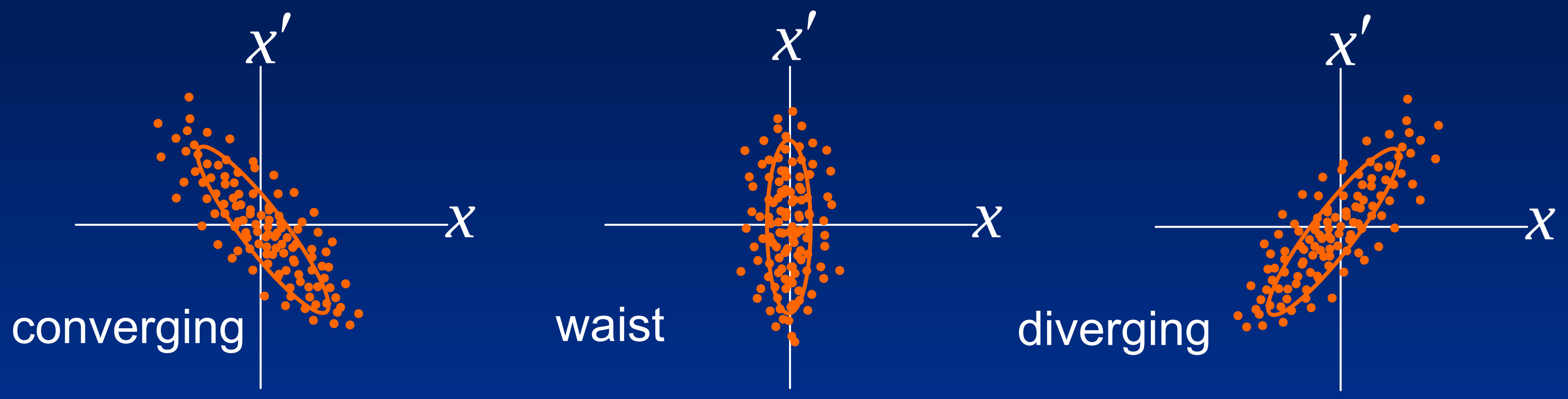
Mathieu-Hill Equations

$$x'' + K_x x = 0 \qquad\qquad y'' + K_y y = 0$$

A system of alternating focusing and defocusing quadrupoles separated by drift space (abbreviated FODO) is used to transport electron beams.



Phase space concept



Beams are treated as a statistical distribution of particles in *x'-x* (also in *y'-y* and γ-*ct*) phase space (trace space, to be exact). We can draw an ellipse around the particles such that 50% of the particles are found within the ellipse. The area of this ellipse is a measure of rms spread of electron distribution in phase space. The rms emittance is area of the ellipse divided by π. Emittance has dimension of length (e.g. microns) since *x'* is dimensionless. Traditionally, emittance has unit of mm-mrad.

$$\boxed{\varepsilon_{rms} = \frac{A}{\pi}}$$

Beam Emittance

Root-mean-square x emittance (for y emittance, replace x with y)

$$\boxed{\varepsilon_{rms,x} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}}$$

Emittance is defined using ensemble averages, denoted by < >, of x^2 and x'^2 and x'-x correlation. The correlation vanishes at the waist (upright ellipse) and rms beam emittance becomes $\sigma_x \sigma_{x'}$ where $\sigma_x = \sqrt{\langle x^2 \rangle}$ is the rms radius in x and $\sigma_{x'} = \sqrt{\langle x'^2 \rangle}$ is the rms spread in x'.

Ensemble average of x^2

$$\langle x^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} \left(x_j - x_0 \right)^2$$

Ensemble average of x'^2

$$\langle x'^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} \left(x'_j \right)^2$$

x'-x Correlation

$$\langle xx' \rangle = \frac{1}{N} \sum_{j=1}^{N} \left(x_j - x_0 \right) x'_j$$

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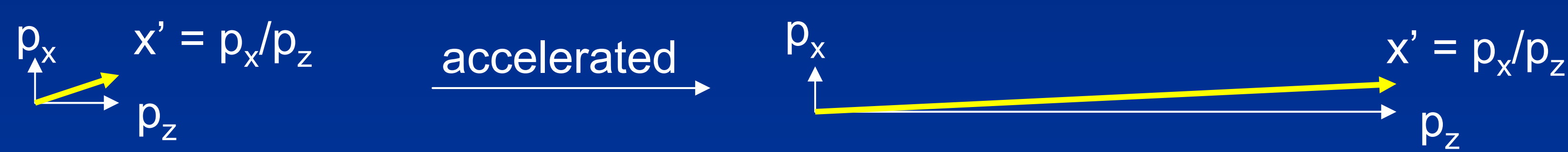
Liouville's Theorem

Liouville's theorem : In the absence of non-linear forces or acceleration, the beam ellipse area in *x*-p_x phase space is conserved. If the forces acting on the beam are linear, its emittance is also conserved.

$$x \cdot p_x = const.$$

If the beam is accelerated, emittance (defined by *x* and *x'*) is not a conserved quantity because *x'* decreases as the axial momentum increases by γ.

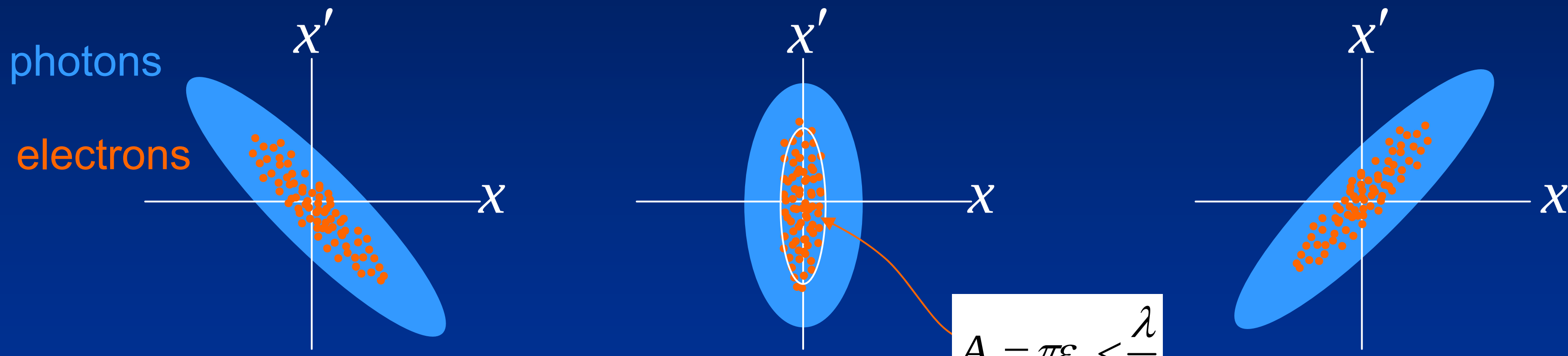


By accelerating the beam (increasing p_z), we reduce the "un-normalized" emittance (also known as Lab frame emittance). The conserved quantity is the **normalized emittance**, un-normalized emittance multiplied by $\beta\gamma$. Normalized emittance is used to specify the quality of electron beams regardless of energy.

$$\boxed{\varepsilon_n = \beta\gamma\varepsilon_u}$$

Electron Beam Emittance Requirement

Electrons' phase-space area must be less than photons' phase space area for efficient energy exchange between electrons and photons



$$A_e = \pi\varepsilon_u < \frac{\lambda}{4}$$

Accelerating the electron beam reduces its un-normalized emittance (adiabatic damping). Beams with large (bad) normalized emittance need to be accelerated to high energy.

$$\varepsilon_u = \frac{\varepsilon_n}{\beta\gamma}$$

At a fixed wavelength and beam energy, the required normalized rms emittance for FEL is

$$\boxed{\varepsilon_n \leq \frac{\gamma\lambda}{4\pi}}$$

Energy Spread Requirement

Electron beam's energy spread must be smaller than the electrons' velocity spread over the interaction length.

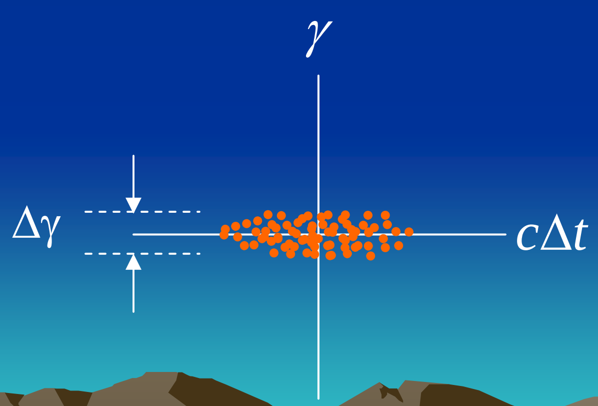
For oscillator FEL, interaction length ~ wiggler length

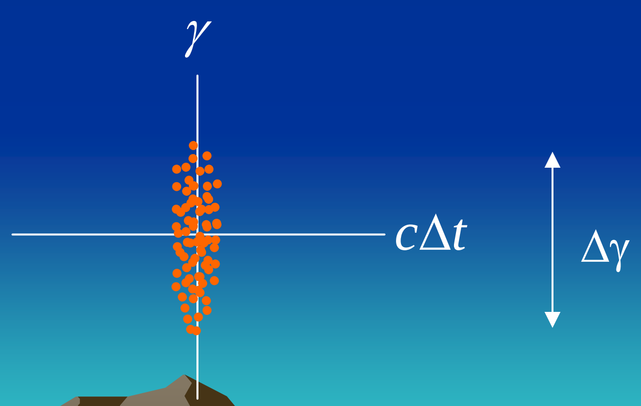
$$\boxed{\frac{\Delta\gamma}{\gamma} \le \frac{1}{2N_w}}$$

For SASE and amplifier FEL, interaction length ~ gain length

$$\boxed{\frac{\Delta\gamma}{\gamma} \le \rho}$$

Uncompressed electron beams have small energy spread and low peak current. Compressed beams have high current and large energy spread.



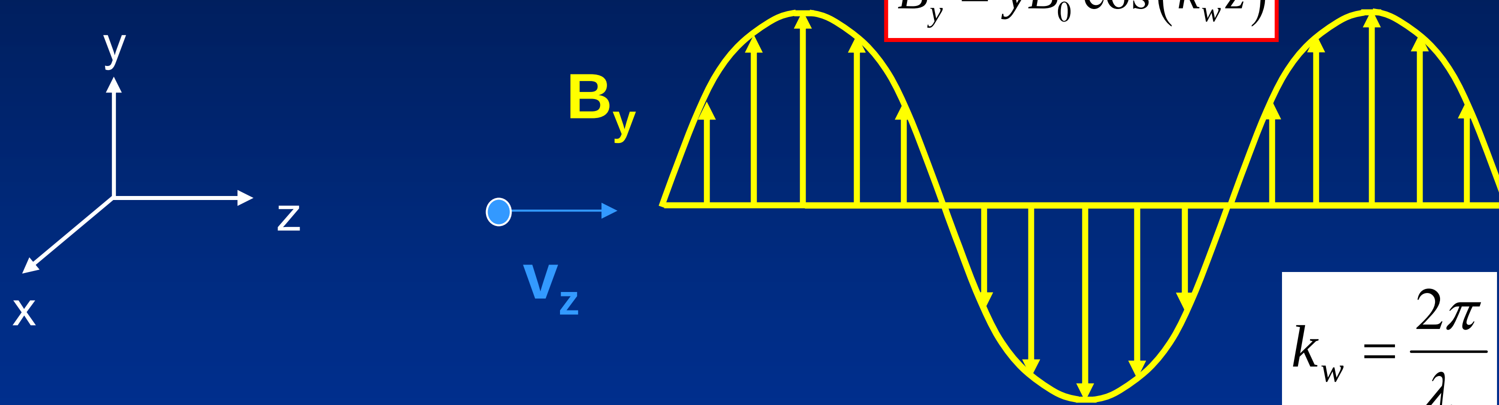


Chapter 3
1-D Theory of FEL

One-dimensional Theory of FEL

- Transverse motion in a wiggler
- Figure 8 motion and harmonics
- Pendulum equation
- FEL bunching
- Bunched beam radiation
- Spontaneous emission spectrum
- Madey's theorem
- Low-gain FEL
- Synchrotron oscillation
- Saturation
- Extraction efficiency
- High-gain FEL
- Self-consistent FEL equations

Equations of Motion



$$\vec{B}_y = \hat{y} B_0 \cos(k_w z)$$

$$k_w = \frac{2\pi}{\lambda_w}$$

Lorentz force laws

$$\boxed{\vec{F}_x = \gamma m_0 \ddot{\vec{x}} = -e(\hat{z}\upsilon_z \times \hat{y}) B_o \cos(k_w z)}$$

$$\vec{F}_z = \gamma m_0 \ddot{\vec{z}} = -e(\hat{x}\upsilon_x \times \hat{y}) B_o \cos(k_w z)$$

For most FEL, v_x is much smaller than v_z. We can ignore the second force equation and consider only motion in x (the wiggle plane).

Equations of Motion (cont'd)

Small-angle approximation: transverse motion is small; axial velocity is almost c

$$\upsilon_x \ll c$$
$$\upsilon_z \approx \beta c$$

Transverse velocity

$$\upsilon_x = \upsilon_z x'$$

Transverse acceleration

$$\ddot{x} = \frac{d}{dt}\left(\upsilon_z x'\right) = \upsilon_z^2 x''$$

Consider only on-axis magnetic field

$$B(z) = B_o \cos\left(k_w z\right)$$

Second derivative of x with respect to z

$$x'' = \frac{d^2 z}{dz^2} = \frac{\ddot{x}}{\upsilon_z^2} \approx \frac{\ddot{x}}{c^2}$$

Lorentz force equation

$$\ddot{x} = \frac{d^2 x}{dt^2} = -\frac{e\upsilon_z}{\gamma m_0} B_o \cos\left(k_w z\right)$$

Rewrite Lorentz force equation in term of second derivative with respect to *z*

$$\boxed{x'' = -\frac{eB_o}{\gamma m_0 c}\cos\left(k_w z\right)}$$

Solution to Transverse EOM

Integrate Lorentz force equation once to obtain deflection angle

$$x' = -\int \frac{eB_o}{\gamma m_0 c}\cos(k_w z)\,dz$$

$$x' = -\frac{eB_o}{\gamma k_w m_0 c}\sin(k_w z) + x'_0$$

$$x' = -\frac{\sqrt{2}a_w}{\gamma}\sin(k_w z) + x'_0$$

x'_0 = initial deflection angle
x_0 = initial position

Integrate again to obtain position

$$x = -\int\left(\frac{\sqrt{2}a_w}{\gamma}\sin(k_w z) + x'_0\right)dz$$

$$x = \frac{\sqrt{2}a_w}{\gamma k_w}\cos(k_w z) + x'_0 z + x_0$$

Transverse motion is periodic with wiggler wavenumber k_w. Wiggler magnetic force is harmonic oscillator's restoring force. Transverse motion in the absence of field errors is given by

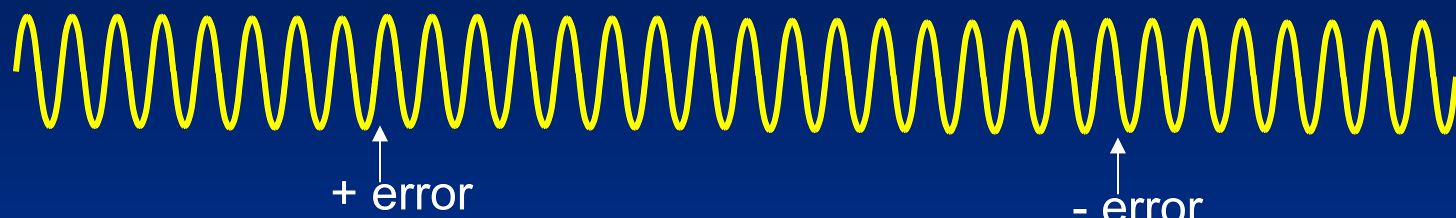
$$\boxed{x'' + k_w^2 x = 0}$$

Wiggler wavenumber

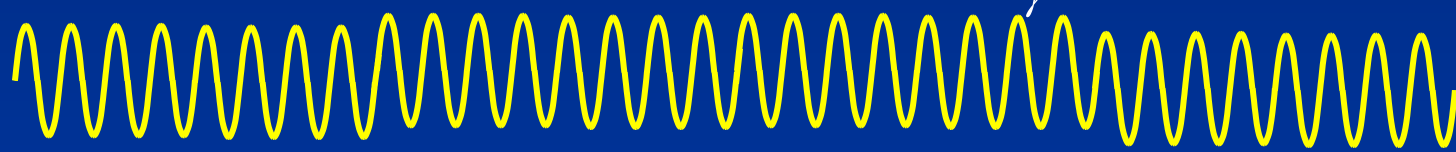
$$k_w = \frac{2\pi}{\lambda_w}$$

B field, deflection and position

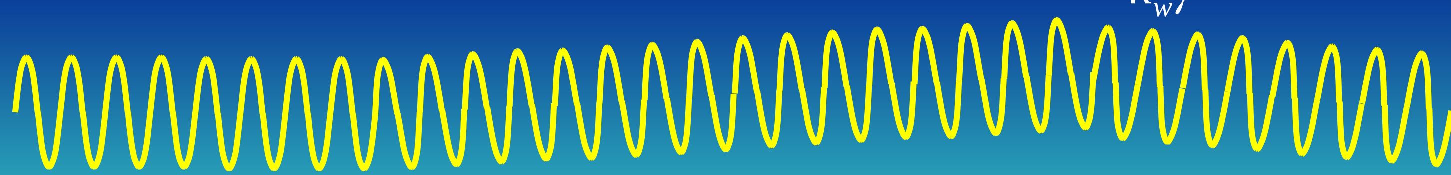
Wiggler magnetic field $B_o \cos(k_w z)$



First integral of field (deflection) $\int B_o \cos(k_w z)\, dz = \dfrac{\sqrt{2} a_w}{\gamma} \sin(k_w z)$



Second integral of field (position) $\int \left(\int B_o \cos(k_w z)\, dz \right) dz = \dfrac{-\sqrt{2} a_w}{k_w \gamma} \cos(k_w z)$



Transverse and longitudinal velocities

Transverse velocity is oscillatory with period equal to the wiggler period

$$\upsilon_x = -\frac{\sqrt{2}ca_w}{\gamma}\sin(k_w z)$$

Longitudinal velocity

$$\upsilon_z^2 = \beta^2 c^2 - \upsilon_x^2$$

$$\upsilon_z^2 = c^2\left(\beta^2 - \frac{2a_w^2}{\gamma^2}\sin^2(k_w z)\right)$$

$$\upsilon_z^2 = c^2\left(1 - \frac{1}{\gamma^2} - \frac{2a_w^2}{\gamma^2}\sin^2(k_w z)\right)$$

Find the square root and use small x approximation $(1 + x)^{1/2} \approx 1 + \frac{1}{2}x$

$$\upsilon_z = c\left[1 - \frac{1}{2\gamma^2}\left(1 + 2a_w^2\sin^2(k_w z)\right)\right]$$

Use sine squared identity

$$2\sin^2(k_w z) = 1 - \cos(2k_w z)$$

Axial velocity oscillates with a period equal to one-half the wiggler period

$$\upsilon_z = c\left(1 - \frac{(1 + a_w^2)}{2\gamma^2} + \frac{a_w^2}{2\gamma^2}\cos(2k_w z)\right)$$

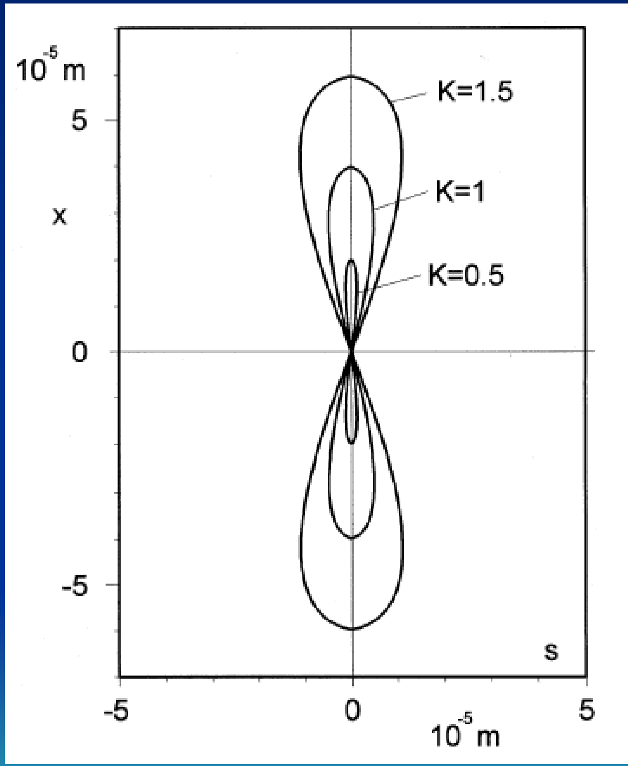
$$\boxed{\upsilon_z = \bar{\upsilon}_z + \frac{ca_w^2}{2\gamma^2}\cos(2k_w z)}$$

Figure 8 Motion

In the reference frame that travels at the electrons' average axial velocity, v_z as given by

$$\bar{\upsilon}_z = c\left(1 - \frac{\left(1 + a_w^2\right)}{2\gamma^2}\right)$$

Motion in reference electron's rest frame



Electrons' transverse and axial motions are coupled. At zero crossing, transverse speed is at a maximum and axial speed a minimum. At the edges, transverse speed is zero and axial speed is at a maximum. Electrons' motion on the x-z plane follows the figure 8.

$$\upsilon_x' = \frac{\sqrt{2}ca_w}{\gamma}\sin\left(k_w' z'\right)$$

$$\upsilon_z' = \frac{ca_w^2}{2k_w'\gamma^2}\cos\left(2k_w' z'\right)$$

Figure 8 motion gives rise to harmonics in spontaneous (incoherent) radiation

Energy exchange between electrons and FEL beam

$$\boxed{\frac{dW}{dt} = -j_\perp \cdot E_s}$$

Transverse electron current

$$j_\perp = ec\dot{x} = -\frac{eca_w}{\gamma}\sin\left(k_w z\right)$$

Plane-wave transverse electric field

$$E_s(z,t) = E_{s,0}\cos(kz - \omega t)$$

$$\frac{d\left(\gamma m_0 c^2\right)}{dt} = -\frac{eca_w E_0}{\gamma}\sin\left(k_w z\right)\cos(kz - \omega t)$$

Rate of energy exchange depends on the phase of the "ponderomotive wave"

$$\frac{d\left(\gamma m_0 c^2\right)}{dt} = -\frac{eca_w E_0}{2\gamma}\sin\left(\left(k_w + k\right)z - \omega t\right)$$

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Resonance Condition

Question: How can an optical wave traveling at the speed of light interact with slower electrons in a fast wave device (e.g., FEL)?

Answer: If the optical wave slips ahead of the electrons exactly one wavelength every wiggler period, the sum of wiggler phase and optical phase is constant, and energy exchange can occur.

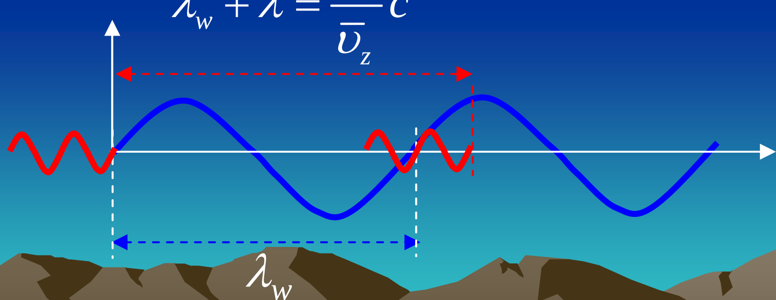
$$\theta = \left(k_w + k\right) z - \omega t = const.$$

$$\frac{d\theta}{dz} = k_w + k - \frac{\omega}{\bar{\upsilon}_z} = 0$$

$$k_w + k = \frac{k}{\left[1 - \dfrac{1+a_w^2}{2\gamma^2}\right]} \approx k + k\left(\frac{1+a_w^2}{2\gamma^2}\right)$$

Resonance wavelength satisfies this condition

$$\lambda_w + \lambda = \frac{\lambda_w}{\bar{\upsilon}_z} c$$



$$k_w = k\left(\frac{1+a_w^2}{2\gamma^2}\right) \rightarrow \boxed{\lambda = \lambda_w\left(\frac{1+a_w^2}{2\gamma^2}\right)}$$

Ponderomotive phase = $-\pi/2$

$k_w z = 0$

$kz - \omega t = -\pi/2$

$(k_w + k)z - \omega t = -\pi/2$

$$\frac{dW}{dt} > 0$$

Electrons gain energy

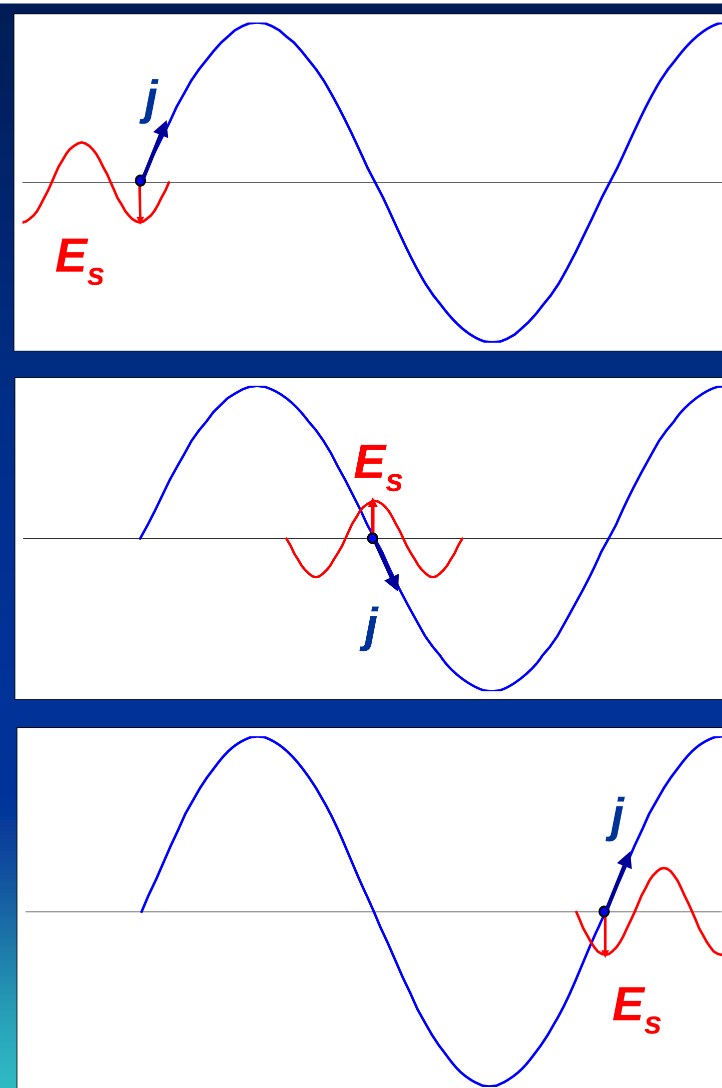
$k_w z = \pi$

$kz - \omega t = -3\pi/2$

$(k_w + k)z - \omega t = -\pi/2$

Electrons gain energy (light is absorbed)

Optical wave slips ahead by λ every λ_w



Ponderomotive phase = 0

$k_w z = 0$

$kz - \omega t = 0$

$(k_w + k)z - \omega t = 0$

$$\frac{dW}{dt} = 0$$

No energy gain or loss

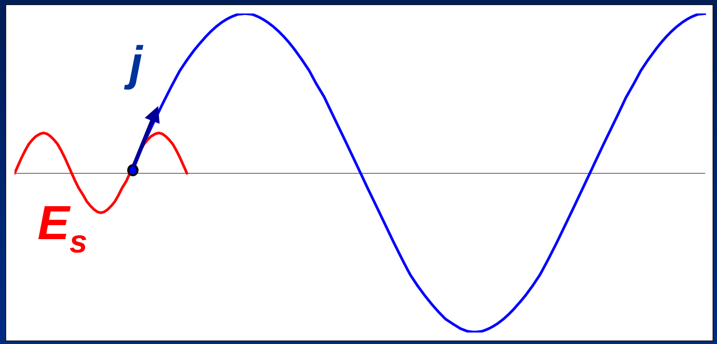
$k_w z = \pi$

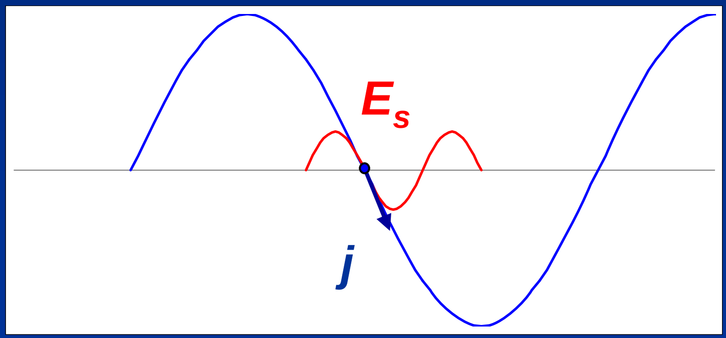
$kz - \omega t = -\pi$

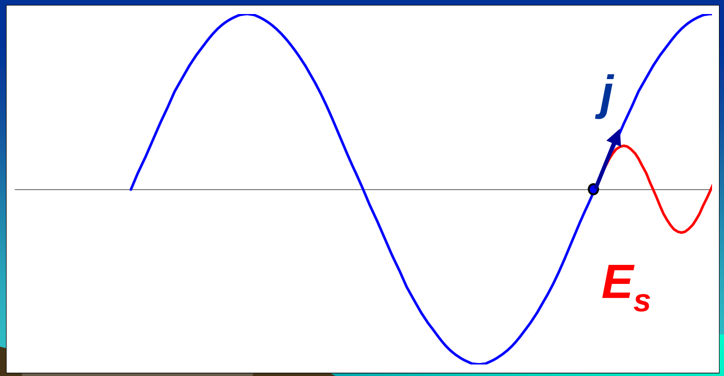
$(k_w + k)z - \omega t = 0$

No energy gain or loss

Optical wave slips ahead one λ







Ponderomotive phase = $\pi/2$

$k_w z = 0$

$kz - \omega t = \pi/2$

$(k_w + k)z - \omega t = \pi/2$

$$\frac{dW}{dt} < 0$$

Electrons lose energy

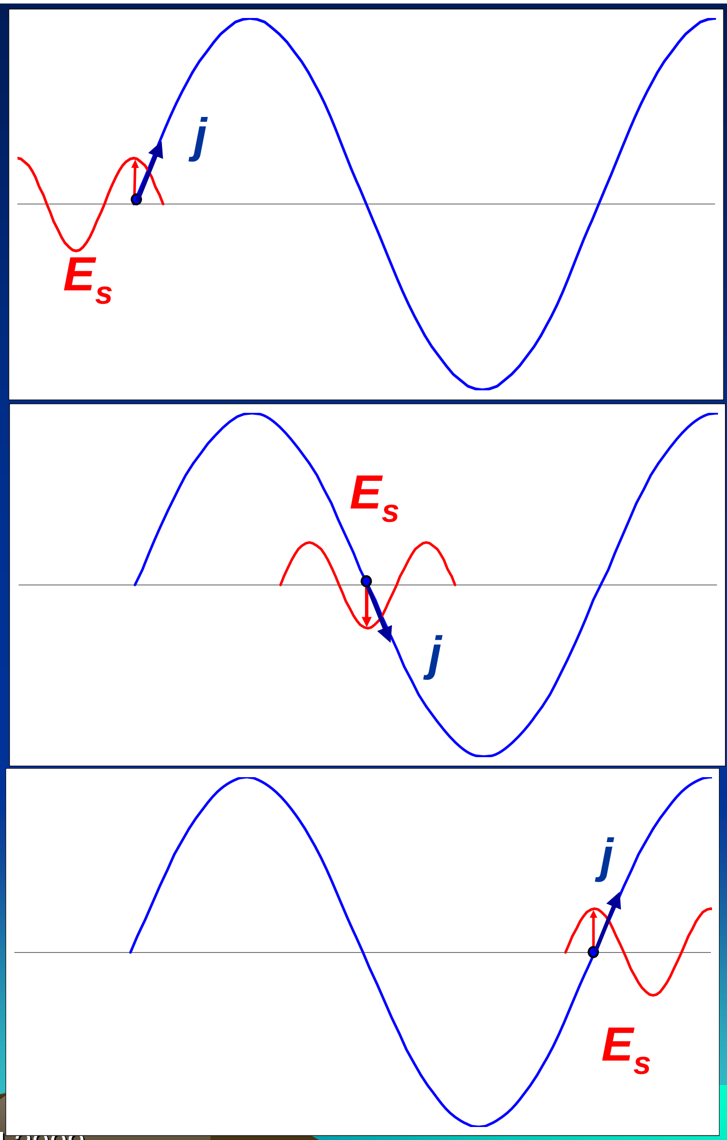
$k_w z = \pi$

$kz - \omega t = -\pi/2$

$(k_w + k)z - \omega t = \pi/2$

Electrons lose energy (FEL gains energy)

Optical wave slips ahead by λ every λ_w



Ponderomotive Wave

The electrons interact with the so-called ponderomotive wave with frequency ω and wavenumber $k_w + k$. The ponderomotive wave is synchronous with the resonant electrons, i.e. those at the zero phase of the ponderomotive wave. The ponderomotive phase velocity, ω divided by $k_w + k$, is slightly less than the speed of light. The phase of the ponderomotive wave is defined by average arrival time of the electrons

$$\theta = \left(k_w + k\right) z - \omega \bar{t}$$

where $k = \dfrac{2\pi}{\lambda}$

Taking derivative with respect to z

$$\boxed{\frac{d\theta}{dz} = \left(k_w + k\right) - \frac{\omega}{\bar{\upsilon}_z}}$$

Average electron axial velocity

$$\boxed{\bar{\upsilon}_z = c\left(1 - \frac{1 + a_w^2}{2\gamma^2}\right)}$$

Phase Equation

Evolution of phase along the wiggler

$$\frac{d\theta}{dz} = (k_w + k) - \frac{\omega}{c}\left(1 + \frac{(1+a_w^2)}{2\gamma^2}\right) = k_w - k\frac{(1+a_w^2)}{2\gamma^2}$$

Using the definition for resonance condition in *k* space $k_w = \frac{k}{2\gamma_R^2}(1+a_w^2)$

$$\frac{d\theta}{dz} = k_w - k_w\frac{\gamma_R^2}{\gamma^2}$$

Define an energy difference relative to the resonant energy γ_R

$$\frac{\Delta\gamma}{\gamma_R} = \frac{\gamma - \gamma_R}{\gamma_R} \ll 1 \longrightarrow \left(\frac{\gamma_R}{\gamma}\right)^2 \approx 1 - 2\left(\frac{\Delta\gamma}{\gamma_R}\right)$$

The phase of individual electrons evolves along the wiggler according to their energy difference relative to the resonance energy

$$\boxed{\frac{d\theta}{dz} = 2k_w\left(\frac{\Delta\gamma}{\gamma_R}\right)}$$

Energy Exchange Equation

Define a dimensionless signal field parameter, a_s

$$a_s = \frac{eE_{s,0}}{km_0c^2}$$

Energy exchange rate depends on the phase of electrons in the ponderomotive potential. Electrons with phase between $-\pi$ and 0 gain energy. Electrons with phase between 0 and π lose energy.

$$\boxed{\frac{d\gamma}{dt} = -\frac{cka_sa_w}{\gamma}\sin\theta}$$

Rewrite the above equation in terms of derivative with respect to z of the energy difference relative to the resonant energy, γ_R

The energy of an electron relative to the resonance energy evolves according to the sine of its phase in the ponderomotive wave

$$\boxed{\frac{d}{dz}\left(\frac{\Delta\gamma}{\gamma_R}\right) = -\frac{ka_sa_w}{\gamma_R^2}\sin\theta}$$

Coupled First-Order Differential Equations

Evolution of relative energy difference and phase along the wiggler

Rate of energy gain/loss along z

$$\frac{d}{dz}\left(\frac{\Delta\gamma}{\gamma_R}\right) = -\frac{k a_s a_w}{\gamma_R^2}\sin\theta$$

Rate of phase change along z

$$\frac{d\theta}{dz} = 2k_w\left(\frac{\Delta\gamma}{\gamma_R}\right)$$

Define new variables, ζ, ν and *a*

$$\zeta = 2k_w\theta$$

$$\nu = \left(\frac{\Delta\gamma}{\gamma_R}\right)$$

$$|a| = \frac{k a_s a_w}{\gamma_R^2} = \Omega^2$$

ζ = angular phase

ν = angular velocity

|a| = height of potential well

Ω = oscillation frequency

Pendulum equations

$$\dot{\nu} = -|a|\sin\zeta$$

$$\dot{\zeta} = \nu$$

Hamiltonian System

Hamiltonian mechanics is useful in representing beam physics because it relies on something being conservative. In the case of a pendulum, the conserved quantity is the total energy of the system of two canonical conjugate variables ν, the angular momentum, and ζ, the angular phase.

Hamiltonian = Total energy

$$\boxed{H = \frac{\nu^2}{2} - |a|\cos\zeta}$$

Kinetic energy

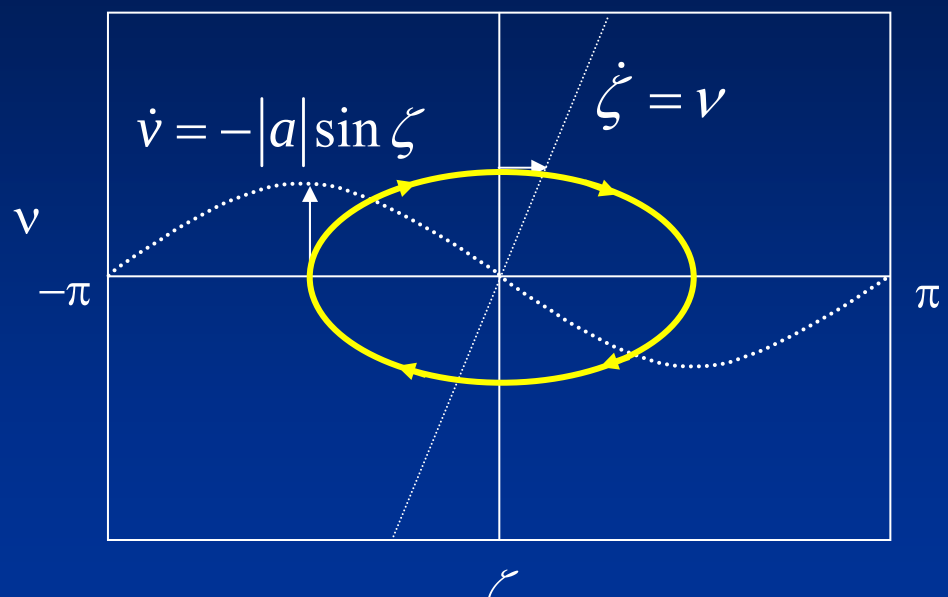
Potential energy

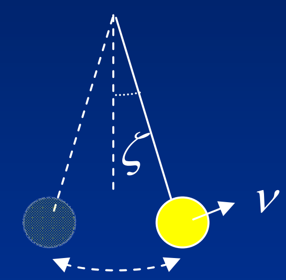
Hamiltonian equations

$$\dot{\nu} = -\frac{\partial H}{\partial \zeta} = -|a|\sin\zeta$$

$$\dot{\zeta} = \frac{\partial H}{\partial \nu} = \nu$$

Pendulum Equations



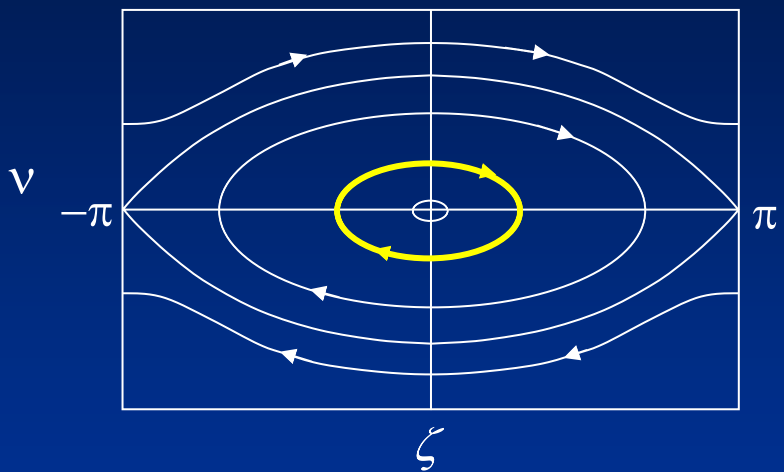


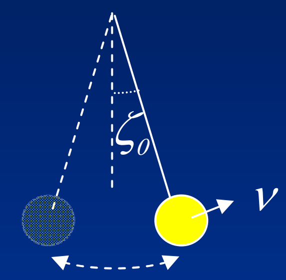
Coupled non-linear 1st order differential equations

$$\dot{v} = -|a|\sin\zeta$$
$$\dot{\zeta} = v$$

Particles rotate clockwise in ν–ζ phase space as the rate of change of ν is proportional to -sin ζ and the rate of change of ζ is ν. Particles follow **elliptical trajectories** each of which corresponds to a constant energy. Higher energies occupy larger ellipses up to phase angle of ± π.

Small-angle Solutions





Small-angle approximation, i.e. $\sin\zeta \sim \zeta$ leads to harmonic solutions with oscillation frequency Ω, square root of |a|

$$\dot{\nu} \approx -\Omega^2 \zeta$$

$$\dot{\zeta} = \nu$$

Second-order differential equation

$$\ddot{\zeta} + \Omega^2 \zeta = 0$$

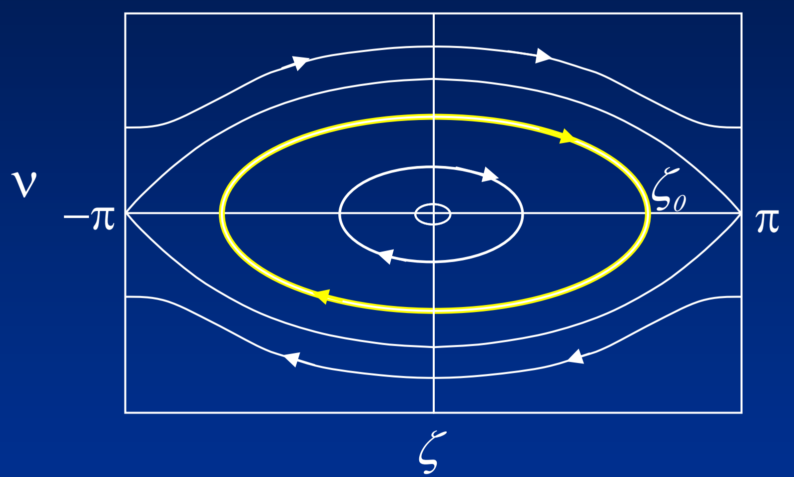
and its solution

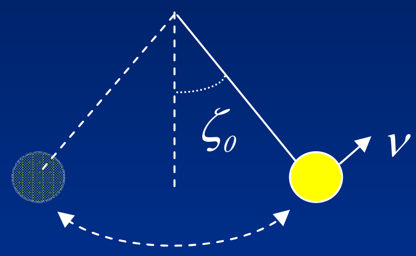
$$\zeta = \sin(\Omega_0 \tau)$$

The small-angle oscillation frequency is known as the synchrotron frequency Ω_0. The synchrotron frequency is proportional to the square root of dimensionless optical field (fourth root of intensity).

$$\boxed{\Omega_0 = \frac{1}{\gamma_R}\sqrt{k a_s a_w}}$$

Large-angle Close-orbit Solutions





Solutions corresponding to **large-angle oscillations** can be solved numerically. The large-angle oscillation frequency is lower than the small-angle synchrotron frequency and approaches zero at $\zeta_0 = \pm\pi$. Oscillation frequency is given by

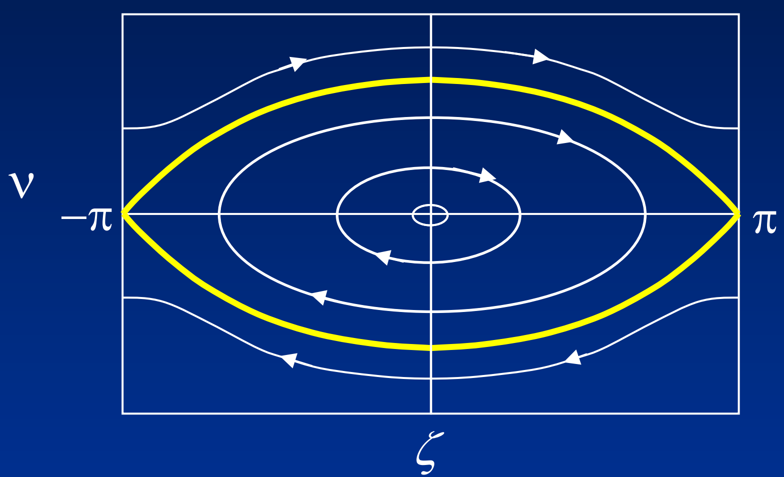
where K : elliptic function.

$$\frac{\Omega}{\Omega_0} = \frac{\frac{\pi}{2}}{K\left(\sin^2\left(\frac{\zeta_0}{2}\right)\right)}$$

Oscillation frequency for initial angle ζ_0 up to $3\pi/4$

$$\boxed{\frac{\Omega}{\Omega_0} \approx 1 - \frac{\zeta_0^2}{16}}$$

Separatrix



Motion at the two nodes, $\zeta = \pm\pi$, vanishes. These are unstable equilibrium points, corresponding to the pendulum at the top. The **separatrix** is the boundary separating trapped and un-trapped trajectories. The region inside the separatrix is called the "**bucket**." The bucket height is proportional to the square root of the optical field (fourth root of optical intensity).

Separatrix for a uniform wiggler

$$\boxed{\nu = \Omega_0 \sqrt{2\left(\cos\zeta + 1\right)}}$$

Bucket half-height

$$\nu_{\max} = \sqrt{\frac{a_s a_w}{1 + a_w^2}}$$

Laser Field and Bucket Height

Dimensionless optical (signal) field parameter, a_s

$$a_s = \frac{eE_{s,0}}{km_0c^2}$$

The electric field of the FEL beam depends on the optical intensity and free space impedance

$$E_{s,0} = \sqrt{2Z_0 I_L}$$

$$Z_0 = 377\Omega$$

Laser intensity depends on power and mode radius

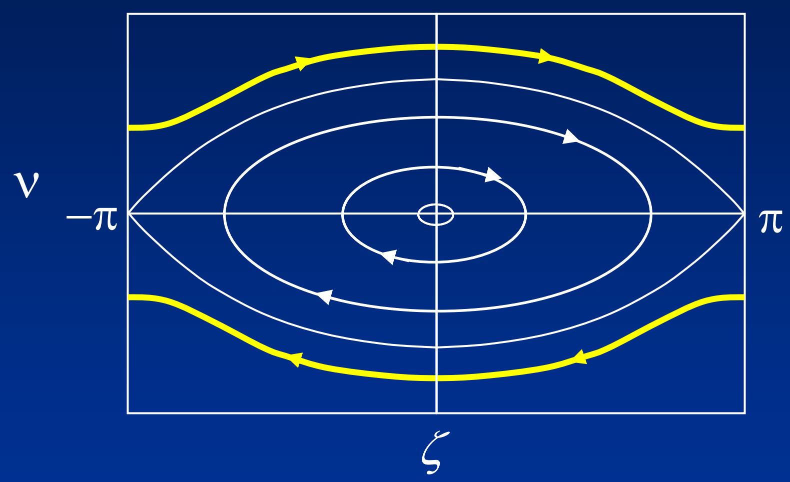
$$I_L = \frac{2P_L}{\pi w_0^2}$$

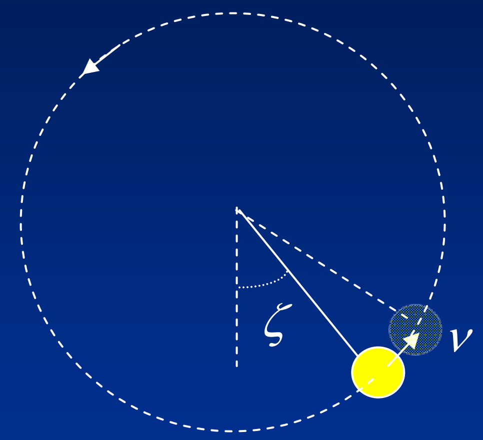
X-ray FEL at 1.5 Å	
Peak power (W)	1.5 x 10^{10}
Intensity (W/cm^2)	5 x 10^{14}
Electric field (V/m)	6 x 10^{10}
a_s	3 x 10^{-6}
ν_{max}	1 x 10^{-3}

Bucket half-height

$$\nu_{max} = \sqrt{\frac{a_s a_w}{1 + a_w^2}}$$

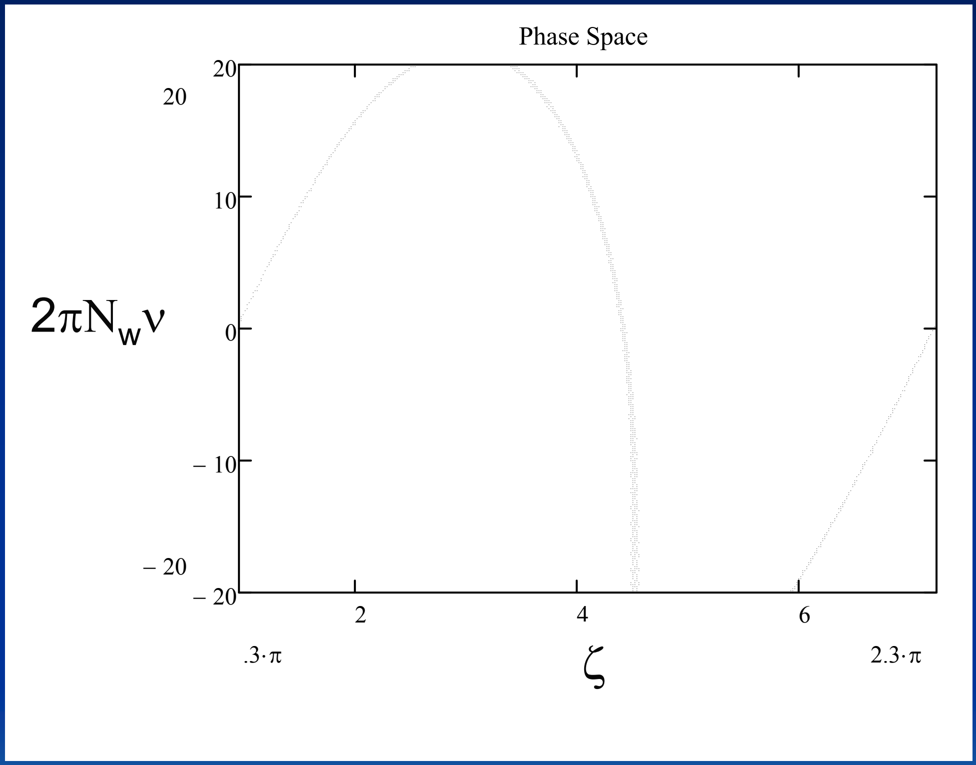
Open Orbits





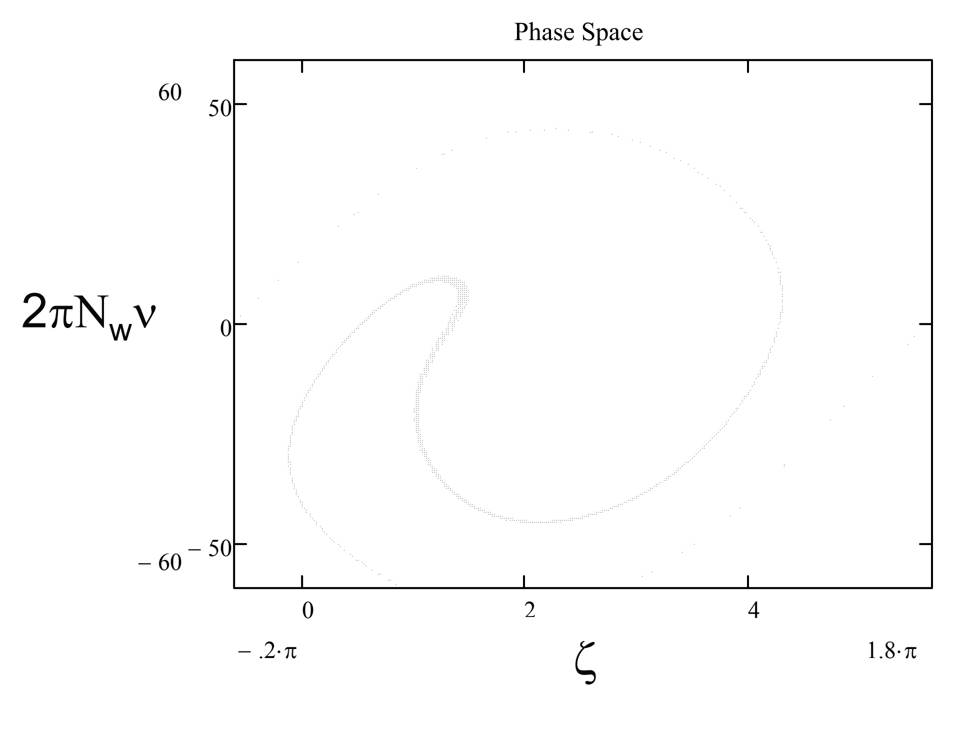
Motion has large angular velocity. The pendulum rolls over the top and librates about the pivot point. The corresponding phase space trajectories are not elliptical. These represent un-trapped electrons outside the "bucket." The un-trapped electrons also provide FEL gain. The electrons at small phases near the top of the "bucket" flow down into the "troughs" and lose energy to the optical field. As the optical field grows, the bucket also grows in height and eventually capture these electrons.

Synchrotron Oscillation Animation



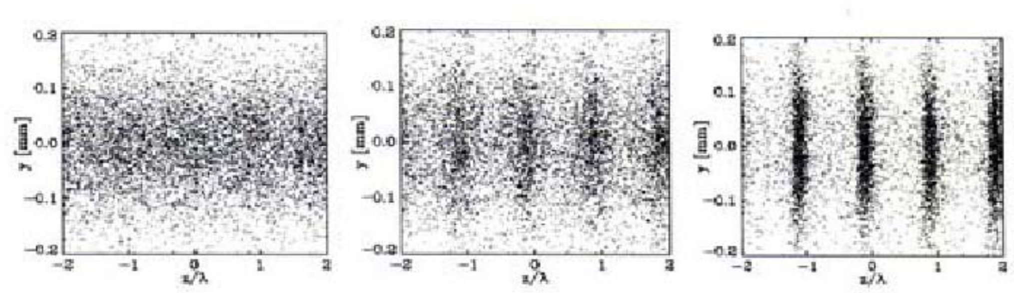
Synchrotron Oscillation Animation

Change scale



Microbunching

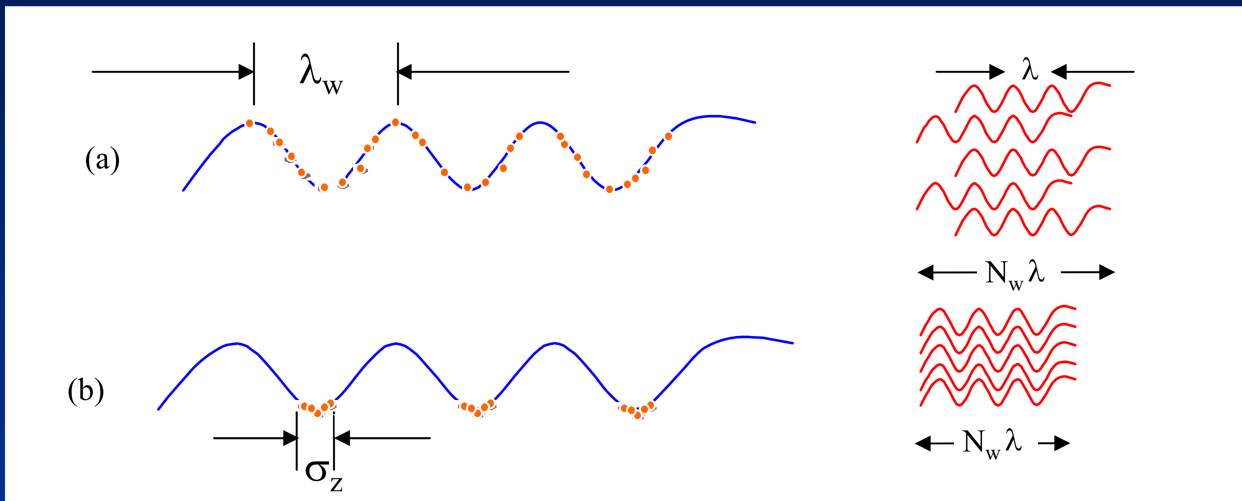
The FEL interaction causes the electrons to gain or lose energy, depending on their ponderomotive phase. Electrons with positive ponderomotive phase lose energy and migrate to the bottom of the bucket. Electrons with negative ponderomotive phase gain energy and move to the top of the bucket. The resulting energy modulation causes the electrons to develop **density modulation** with period of the radiation wavelength. The bunched electrons radiate higher power, i.e. it amplifies the electromagnetic wave. As the electric field of the electromagnetic wave increases, the height of the bucket also increases. When the electrons are completely bunched, FEL power is saturated. Microbunching is responsible for harmonic generation (the Fourier transform of short bunches has high frequency components).



At entrance to the undulator | Exponential gain regime | Saturation(maximum bunching)

Courtesy of S. Reiche

Radiation from bunched beam



$$|E| \propto \sqrt{N_e}$$

$$I_{incoherent} \propto N_e$$

$$|E| \propto N_e$$

$$I_{coherent} \propto N_e^2$$

Electrons at the wiggler entrance are randomly distributed (a). Randomly distributed electrons radiate incoherently, i.e. the electric fields of N_e randomly distributed wave trains with $N_w\lambda$ (N_w is the number of wiggler periods and λ is the wavelength) add incoherently. The total electric field is proportional with square root of N_e. The **spontaneous radiation intensity scales with N_e**.

Near saturation, the electrons are bunched into microbunches with bunch length σ_z less than radiation wavelength (b). The electric fields of N_e wave trains scales with N_e, and the **coherent radiation intensity scales with N_e^2**.

Spontaneous Emission

Spectral and angular energy fluence of spontaneous emission radiation from a planar wiggler as a function of frequency detuning from resonance condition

$$\frac{dW}{d\omega d\Omega} = \frac{e^2\gamma^2 N_w^2 N_e}{2\pi\varepsilon_0 c}\left[JJ(a_w)\right]^2\left(\frac{a_w}{1+a_w^2}\right)^2\left(\frac{\sin \Delta/2}{\Delta}\right)^2$$

Difference between J_0 and J_1 Bessel functions

$$JJ(\xi) = J_0(\xi) - J_1(\xi)$$

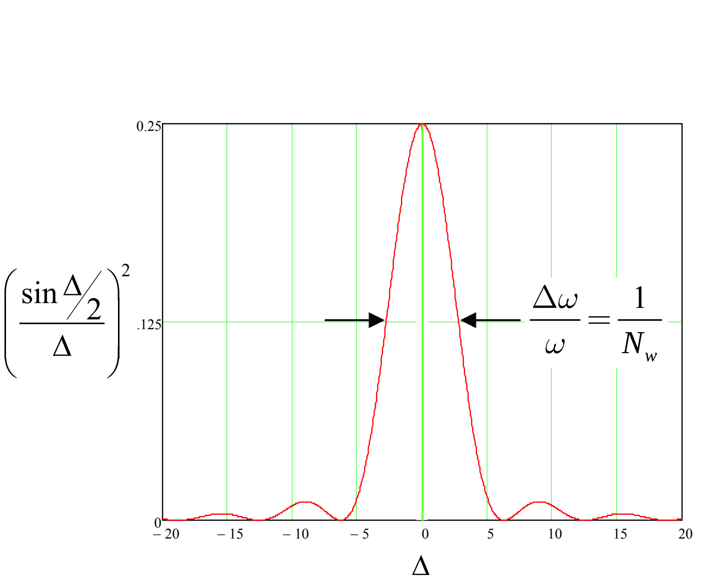
$$\xi = \frac{a_w^2}{2\left(1+a_w^2\right)}$$

Approximation for small ξ

$$\left[JJ(\xi)\right] \approx 1 - \frac{\xi}{2} - \frac{\xi^2}{4}$$

Frequency detuning

$$\Delta = 2\pi N_w \frac{\Delta\omega}{\omega}$$



Spontaneous emission is peaked at zero detuning (resonant wavelength)

Spontaneous Emission (cont'd)

Consider only photons within coherent spectral bandwidth and solid angle

Coherent spectral bandwidth

$$\frac{\Delta\omega}{\omega} = \frac{1}{N_w}$$

Coherent angle

$$\theta = \sqrt{\frac{\lambda}{L_w}}$$

Solid angle

$$\pi\theta^2 = \frac{\pi\lambda}{N_w\lambda_w}$$

Number of coherent spontaneous photons per electron does not depend on N_w

$$\boxed{\frac{N_{photon}}{N_e} = \pi\alpha\left[JJ(a_w)\right]^2 \left(\frac{a_w}{1+a_w^2}\right)^2}$$

where α = fine structure constant

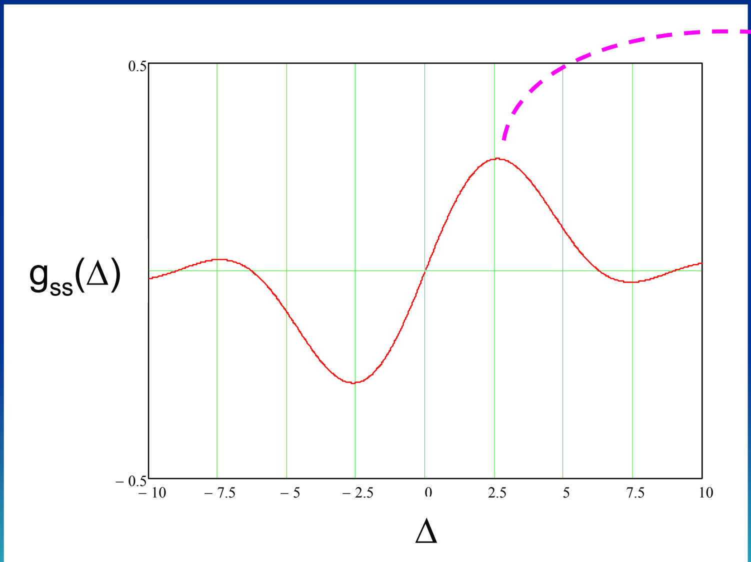
$$\alpha = \frac{e^2}{\hbar c 4\pi\varepsilon_0} \approx \frac{1}{137}$$

For typical values of a_w, on average we need 200 electrons to generate 1 spontaneous photon within coherent angle and bandwidth

Madey's Theorem

Madey's Theorem: The small-signal gain spectrum (gain versus energy detuning) for a low-gain FEL is the derivative of the spontaneous emission spectrum. The small-signal gain is positive (amplification) at positive detuning, zero on resonance and negative (absorption) at negative detuning.

$$\boxed{g_{ss}(\Delta) = \frac{4(4\pi\rho N_w)^3}{\Delta^3}\left(1 - \cos\Delta - \frac{\Delta}{2}\sin\Delta\right)}$$



Maximum gain is at $\Delta = 2.6$

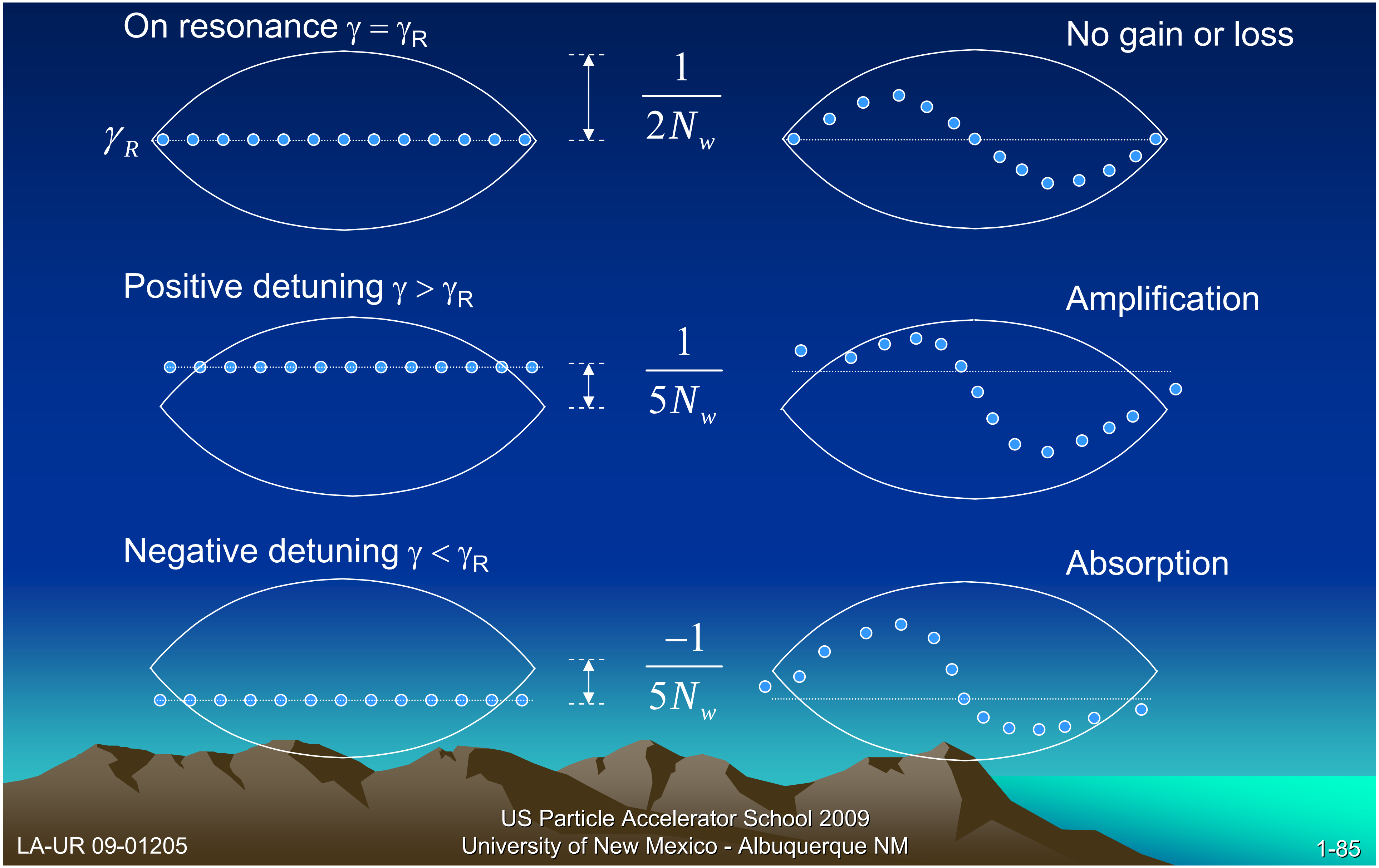
$$\Delta = 4\pi N_w\left(\frac{\Delta E}{E}\right)$$

$$\boxed{\frac{\Delta E}{E} = \frac{2.6}{4\pi N_w} \approx \frac{1}{5N_w}}$$

Maximum gain occurs at positive energy detuning (higher energy) than resonance, or at a fixed energy, longer wavelength.

$$\boxed{g_{ss}^{\max} \approx 2(2\pi\rho N_w)^3}$$

Visualization of Madey's Theorem



Small-Signal Gain

The small-signal gain for a planar wiggler at the peak of the gain curve, assuming the electron beam radius σ is smaller than the optical beam, is

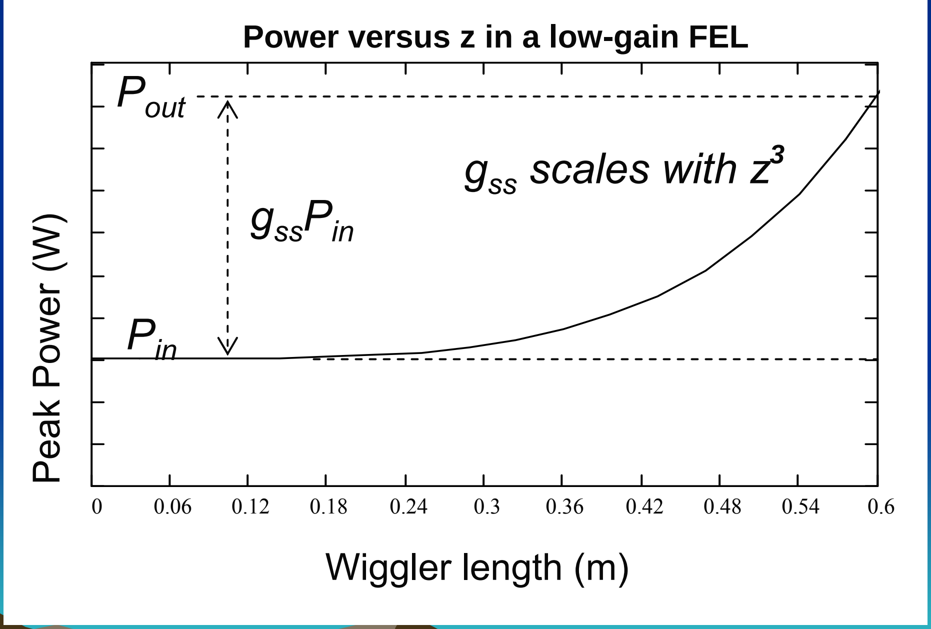
$$g_{ss} = \left(\frac{2\pi N_w}{\gamma}\right)^3 \left(\frac{[JJ]a_w}{\sigma k_w}\right)^2 \left(\frac{I}{I_A}\right)$$

where $k_w = \dfrac{2\pi}{\lambda_w}$

and I_A (Alfven current) = 17 kA

Small-signal gain in a low-gain FEL is proportional to N_w^3

$$P_{out} = (1 + g_{ss})P_{in}$$



Large-Signal Gain

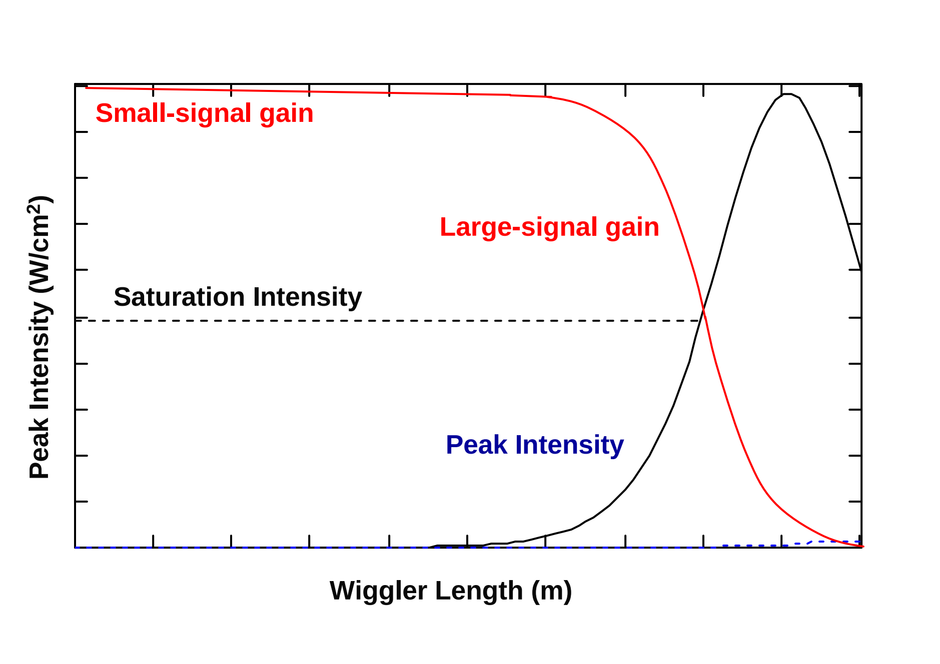
FEL gain is reduced when optical intensity approaches the saturation intensity,

$$I_S = \frac{1}{8\pi}\frac{mc^3}{\sigma}\left(\frac{1}{[JJ]a_w\lambda_w}\right)^2\left(\frac{\gamma}{N_w}\right)^4$$

Large-signal gain

$$g(I) = \frac{g_{ss}}{1+\left(\frac{I}{I_S}\right)}$$

At high intensity, more electrons reside at the bottom of the bucket and FEL gain decreases. Saturation intensity is the intensity at which FEL gain reduces to one-half of g_{ss}.



Synchrotron Oscillation

Energy and phase equations

$$\frac{d}{dz}\left(\frac{\Delta\gamma}{\gamma_R}\right) = -\frac{k a_s a_w}{\gamma_R^2}\sin\theta$$

$$\frac{d^2\theta}{dz^2} = 2k_w \frac{d}{dz}\left(\frac{\Delta\gamma}{\gamma_R}\right)$$

2nd-order differential equation of phase evolution with z

$$\frac{d^2\theta}{dz^2} + K_S^2 \sin\theta = 0$$

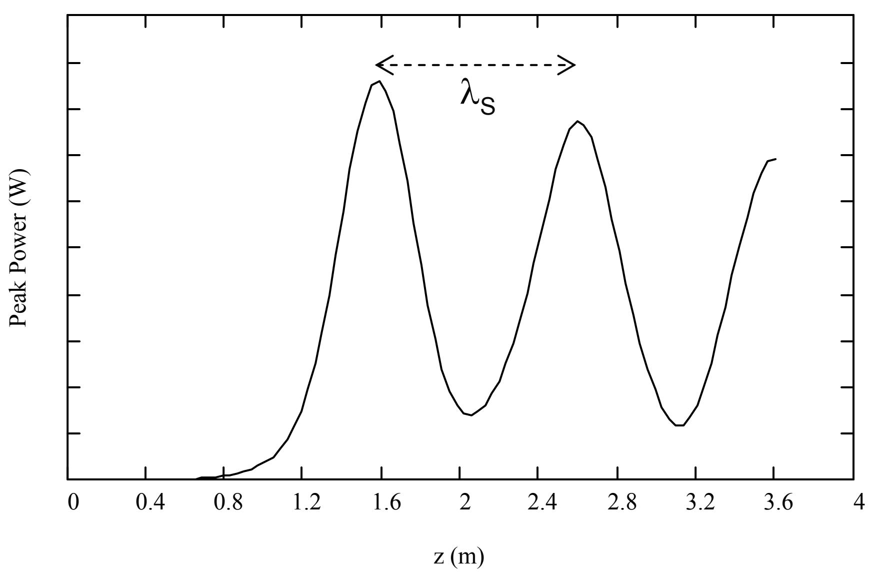
Synchrotron oscillation wavenumber

$$K_S = \sqrt{\frac{2k_w k a_s a_w}{\gamma_R^2}} = 2k_w\sqrt{\frac{a_s a_w}{\left(1+a_w^2\right)}}$$

Synchrotron period

$$\boxed{\lambda_S = \frac{\lambda_w}{2}\sqrt{\frac{1+a_w^2}{a_s a_w}}}$$

Plot of power vs z showing synchrotron oscillations



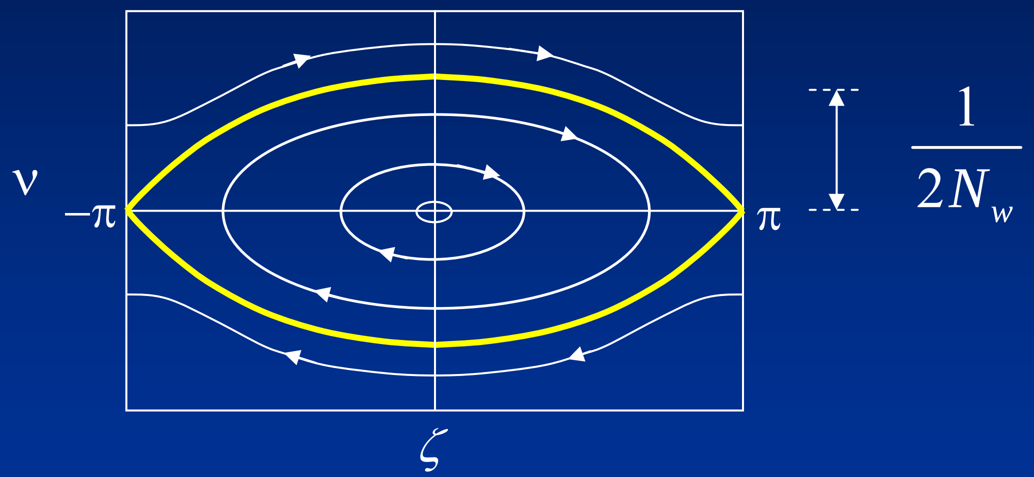
Extraction Efficiency

Wiggler length ~ synchrotron period

$$L_w \approx \lambda_S = \frac{\lambda_w}{2}\sqrt{\frac{1+a_w^2}{a_s a_w}}$$

$$L_w = \frac{\lambda_w}{2\nu_{max}}$$

$$\nu_{max} = \frac{\lambda_w}{2L_w}$$



At saturation, the wiggler length is about the same as a synchrotron oscillation period. The electrons rotate to the bottom of the "bucket." The bucket half-height is inversely proportional to $2N_w$.

$$\boxed{\nu_{max} = \frac{1}{2N_w}}$$

High-Gain FEL

Dimensionless Pierce parameter as a function of k_w (left) or λ_w (right)

$$\rho = \frac{1}{2\gamma}\left(\frac{[JJ]a_w}{\sigma k_w}\right)^{\frac{2}{3}}\left(\frac{I}{I_A}\right)^{\frac{1}{3}}$$

$$\rho = \frac{1}{\gamma}\left(\frac{[JJ]a_w\lambda_w}{4\sqrt{2}\pi\sigma}\right)^{\frac{2}{3}}\left(\frac{I}{I_A}\right)^{\frac{1}{3}}$$

Recall JJ is the difference between J_0 and J_1 Bessel functions of argument ξ

$$[JJ] = J_0(\xi) - J_1(\xi)$$

$$J_0(\xi) \approx 1 - \frac{\xi^2}{4} \qquad J_1(\xi) \approx \frac{\xi}{2}$$

$$[JJ] \approx 1 - \frac{\xi}{2} - \frac{\xi^2}{4}$$

where

$$\xi = \frac{a_w^2}{2\left(1 + a_w^2\right)}$$

High gain FEL is applicable in a long wiggler driven by a high-brightness electron beam (one with high peak current and small emittance). The wiggler length must be significantly longer than the power gain length, given by

Power gain length

$$L_G = \frac{\lambda_w}{4\pi\sqrt{3}\rho}$$

Power Growth in High-Gain FEL

Power vs distance

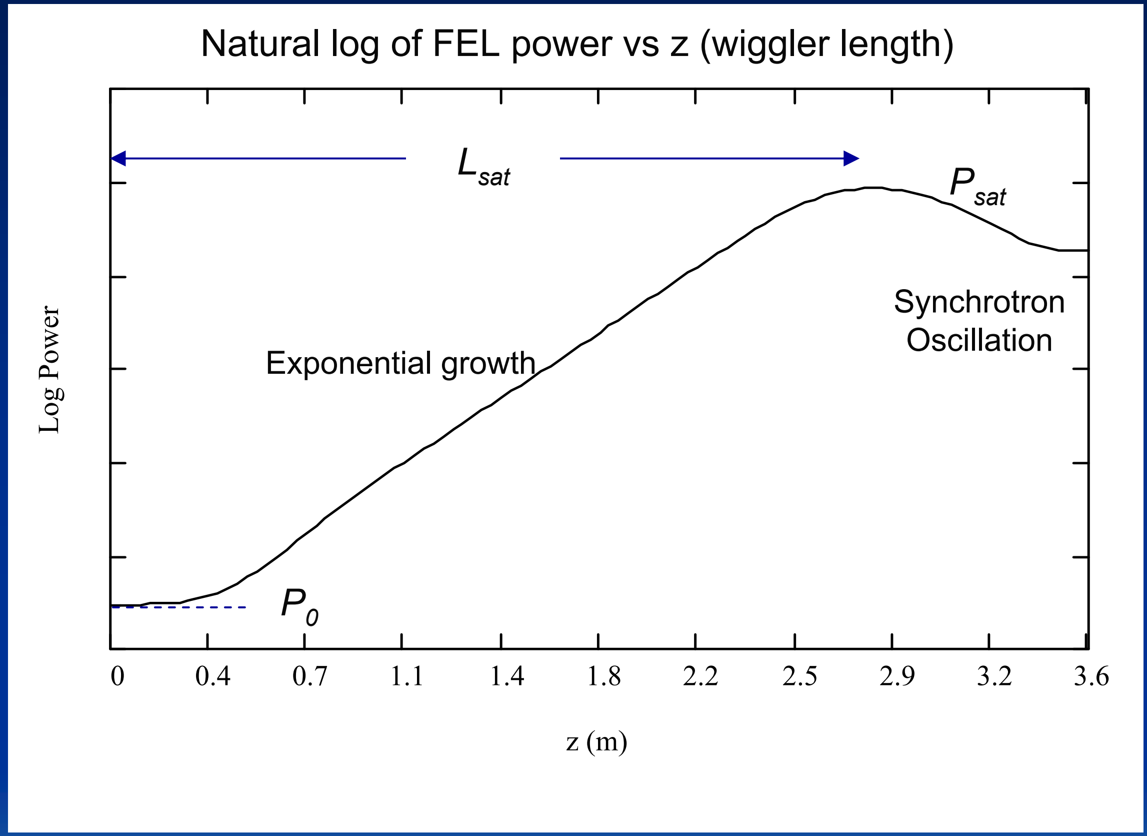
$$P(z) = \frac{1}{9} P_0 \exp\left(\frac{z}{L_G}\right)$$

Saturation length

$$L_{sat} = L_G \ln\left(\frac{9P_{sat}}{P_0}\right)$$

Saturation power

$$P_{sat} \approx \rho \frac{IE_b}{e}$$



Power grows exponentially with distance by one e-folding (2.7) every power gain length. Starting from noise, the FEL saturates in 20 power gain lengths. FEL saturation power, P_{sat}, is approximately ρ times the electron beam power.

Slowly Varying Envelope Approximation

So far, we've considered only the electron phase-space motion. To be complete, we must write self-consistent FEL equations for *N* electrons and the optical field. We'll treat the optical field as a slowly varying phasor (ignoring the optical frequency oscillation). The phasor's amplitude is the usual dimensionless optical field a_s. This is known as the **Slowly Varying Envelope Approximation** (SVEA).

Optical electric field with fast oscillations

$$E(t) = E_0 \exp\left[i(kz - \omega t + \phi)\right]$$

SVEA phasor

$$a = a_s \, e^{-i\phi_s}$$

Wave equation without the fast time scale terms (e.g. 2nd order derivatives)

$$\left[\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right] a = \frac{(4\pi)^3 [JJ]}{2k\Sigma_b}\left(\frac{I}{I_A}\right)\left[a_w \left\langle \frac{e^{-i\theta}}{\gamma} \right\rangle - ia \left\langle \frac{1}{\gamma} \right\rangle\right]$$

The electron bunch is assumed to be many wavelengths long, so the beam current density is assumed to be independent of z over many wavelengths.

Self-Consistent FEL Equations

Evolution of the *j*th electron's phase and energy

$$\frac{d\theta_j}{dz} = k_w - \frac{k}{2\gamma_j^2}\left[1 + a_w^2 - 2a_w a_s\left[JJ\right]\cos\theta_j\right]$$

$$\frac{d\gamma_j}{dz} = -\frac{k a_w a_s}{\gamma_j}\left[JJ\right]\sin\theta_j$$

Evolution of optical phasor's phase and amplitude

$$\frac{d\phi_s}{dz} = \frac{(4\pi)^3}{2k\Sigma_b}\left(\frac{I}{I_A}\right)\left[\frac{a_w\left[JJ\right]}{a_s}\left\langle\frac{\cos\theta}{\gamma}\right\rangle - \left\langle\frac{1}{\gamma}\right\rangle\right]$$

$$\frac{da_s}{dz} = \frac{(4\pi)^3 a_w\left[JJ\right]}{2k\Sigma_b}\left(\frac{I}{I_A}\right)\left\langle\frac{\sin\theta}{\gamma}\right\rangle$$

The <cosθ> term corresponds to the real part of the e-beam's susceptibility (refractive index) and <sinθ> term corresponds to the imaginary part (gain).

Scaled Variables

- Scaled axial position

$$\tau = \frac{z}{L_w}$$

- Dimensionless current density

$$\bar{j} = 2\left(4\pi N_w \rho\right)^3$$

- Scaled phasor equation

$$\frac{da}{d\tau} = -\bar{j}\left\langle e^{-i\theta}\right\rangle$$

$$\frac{da}{d\tau} = \frac{\bar{j}}{2}\int_0^{\tau} \tau' a\left(\tau - \tau'\right) e^{-i\nu_0 \tau'} d\tau'$$

Cubic Equation

Take the derivative of the last equation successively

$$\frac{d^3a(\tau)}{d\tau^3} = -\frac{\bar{j}a(\tau)}{2}$$

Assuming solutions are of the form $e^{i\lambda\tau}$ and at resonance condition, we obtain the characteristic **cubic dispersion relation**

$$\lambda^3 + \frac{\bar{j}}{2} = 0$$

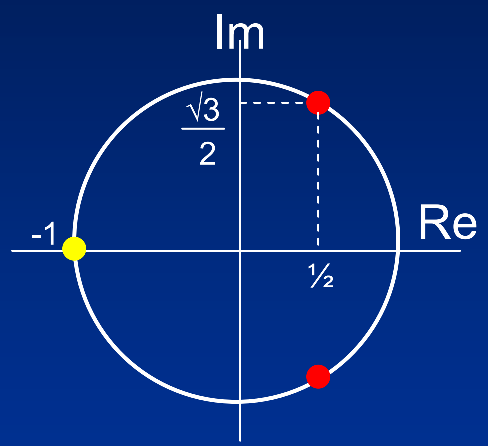
Note: λ are roots of the cubic equation, not wavelength

Solutions of the cubic equation are of the form

$$a(\tau) = a_0 e^{i\lambda\tau}$$

Solutions to Cubic Equation

Three roots of the cubic equation



Complex root

$$\lambda_1 = \left(\frac{\bar{j}}{2}\right)^{\frac{1}{3}} \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

Complex root

$$\lambda_2 = \left(\frac{\bar{j}}{2}\right)^{\frac{1}{3}} \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$$

Real root

$$\lambda_3 = -\left(\frac{\bar{j}}{2}\right)^{\frac{1}{3}}$$

Solutions in electric field

$$E(\tau) = \frac{E_0 e^{\frac{i\tau}{2}\left(\frac{\bar{j}}{2}\right)^{\frac{1}{3}}}}{3} \left(\underset{\text{growing mode}}{e^{\left(\frac{\bar{j}}{2}\right)^{\frac{1}{3}}\left(\frac{\tau\sqrt{3}}{2}\right)}} + \underset{\text{decaying mode}}{e^{\left(\frac{\bar{j}}{2}\right)^{\frac{1}{3}}\left(\frac{-\tau\sqrt{3}}{2}\right)}} + \underset{\text{oscillatory mode}}{e^{\left(\frac{\bar{j}}{2}\right)^{\frac{1}{3}}\left(\frac{-i3\tau}{2}\right)}} \right)$$

Exponential Growth

In the limit of large z, only the growing mode needs to be considered. The optical field vs scaled length τ is given by

$$E(\tau) = \tfrac{1}{3} E_0 e^{\frac{i\tau}{2}\left(\frac{\bar{j}}{2}\right)^{\frac{1}{3}}} e^{\frac{\tau\sqrt{3}}{2}\left(\frac{\bar{j}}{2}\right)^{\frac{1}{3}}}$$

Multiplying the electric field by its complex conjugate yields the FEL intensity versus the scaled length τ

$$|E(\tau)|^2 = \frac{|E_0|^2}{9}\exp\left(\tau\sqrt{3}\left(\frac{\bar{j}}{2}\right)^{\frac{1}{3}}\right)$$

Plug in the expressions for τ and *j*, we arrive at the expression for intensity vs. distance in the wiggler. This equation gives the exponential growth with wiggler length and the initial 1/9 reduction in signal intensity.

$$\boxed{I(z) = \frac{I_0}{9}\exp\left(\frac{4\pi\sqrt{3}\rho z}{\lambda_w}\right)}$$

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